이학박사학위논문

# New methods to count instantons

## 새로운 방법을 통한 순간자 계산

2020 년 2 월

서울대학교 대학원

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이 논문을 이학박사 학위논문으로 제출함

2020 년 월

서울대학교 대학원

물리천문학부 물리학전공

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# Abstract

In this thesis we introduce new methods to count instantons in 4,5 and 6 dimensions. When the gauge group is classical simple Lie group, instanton moduli space can be understood by ADHM formalism which are a matrix theory with auxiliary(gauge) group and its 1d or 2d gauged extension on the worldline/sheet of an instanton particle(5d)/string(6d).

However there are difficulties in understanding instanton moduli by the lack of known ADHM construction when the gauge group is exceptional or matters are in exotic representations. Among such cases, one may compute instanton partition functions by extending ADHM formalism of classical gauge group. With this method we also compute partition function of self-dual strings in 6d $\mathcal{N} = (1,0)$ non-Higgsable cluster SCFTs.

On the other hand one can obtain the so-called "blowup equation" which is the consistency condition that the partition function satisfies by blowing-up the center of instanton in $\mathbb{R}^4$. Using blowup equations, one can recursively compute the instanton partition function for general gauge group and matters by knowing the perturbative partition function. In the later part of the thesis we first derive the blowup equations of 4d $\mathcal{N} = 2$ partition functions, find analogous blowup equations in 5d, surveying when they satisfy, and finaly compute the instanton partition functions of general gauge group and matters using them.



**Keywords**: Instantons, supersymmetry, SCFT, partition function, string theory

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# Chapter 1

# Introduction

Quantum field theory (QFT) has been the most useful framework to describe the microscopic behavior of our nature. It was employed in describing all kind of quantum phenomena such as particle physics, superconductivity, quantum hall effect, etc. Many of this success was accomplished with the help of perturbation theory. Perturbation theory enables one to understand weakly interacting system from the non-interacting system corrected in series of small coupling constant. On the other hand, by definition perturbation theory can't be used to investigate the strongly interacting system such as quarks forming baryons. Since the couplings run via renormalization group (RG) flow, understanding such strong coupling dynamics is crucial to study the phases of QFT.

Many successful studies on strong coupling physics was achieved with supersymmetries that regulate the quantum corrections and make theory simpler. Especially 4d $\mathcal{N} = 2$ theories provided simple enough, yet interesting phenomena had been discovered through the Colomb phase analysis from the study by

Seiberg and Witten [2] such as discoveries of interacting superconformal field theories (SCFTs) without Lagrangian desecription, dualities or some integrability. It guided many other studies on strong coupling dynamics and also provided better understanding on both quantum field theories and string theories.

In higher dimensions ($d > 4$) strong coupling dynamics become even more crucial. For ages from the birth of QFT people had believed that no UV complete gauge theory can be formulated in $d > 4$ because the gauge coupling is non-renormalizable. It may not be the case when there exist some UV fixed point in RG flow [84,85]. In such cases the gauge theory describes the IR physics flowed from the UV fixed point by finite gauge coupling. Thus non-perturbative studies on effective gauge theories in strong coupling regime provide some clues on the UV fixed point. Indeed, any particularly known UV fixed points are predicted from string theory. They are either formulated on the world volume of certain branes, or from string theory compactified on some Calabi-Yau manifold. They inherit the supersymmetry (SUSY) from string theory, becoming SCFTs. Their superconformal symmetry exists only at $d \leq 6$. In 5d there exist only one kind of SCFTs called $\mathcal{N} = 1$ SCFTs. On the other hand in 6d, there are two kinds of possible SCFTs called $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (2, 0)$ SCFTs respectively. In this thesis we will pay attention to minimal SUSY theories with 8 supercharges.

On the Coulomb branch physics in $d \geq 4$, instantons govern the non-perturbative effects. Instantons are solitonic objects whose action, mass or tension is proportional to (gauge coupling)$^{-2}$. In the supersymmetric gauge theories instantons preserve half of the supersymmetry, called $\frac{1}{2}$-BPS objects. Instanton partition functions which count the $\frac{1}{2}$-BPS instantons become a usefull observable of SCFTs lying on RG fixed points. In this thesis, we review the traditional technique and introduce two different new techniques to compute the instanotn partition functions. For classical gauge groups with hypermultiplets in some specific representations, instanton moduli dynamics are described by the supersymmetric gauge theory defined on their worldvolume. It is basically the extension of the matrix moduli space of 4d Yang-Mills theory. On the other hand, when the gauge group is exceptional one or the matters are in some other representations, for example matters in $SO(N)$ spinor representation, the moduli dynamics are not generally known. We introduce other methods to compute instanton partition functions that are applicable in such cases.

The remaining chapters consist of followings. In chapter 2, we review the 4,5 and 6 dimensional field theories with 8 supercharges. We first explain the 4d $\mathcal{N}=2$ gauge theories and their Coulomb moduli space, which is essential setting of instanton counting. Also we introduce classifications of 5d from Coulomb branch analysis and M-theory compactified on Calabi-Yau 3-folds. Similarly classification of 6d $\mathcal{N}=(1,0)$ SCFTs by compatifying F-theory on elliptic fibered Calabi-Yau 3-folds. Through the SCFT classification, we list what kind of higher dimensional gauge theories can be consistentely defined. In chapter 3, we review the classical analysis of instantons and their moduli spaces. Starting from those of 4d pure Yang-Mills instantons, we analogously introduce how the instanton moduli dynamics is described on their worldvolume. In chapter 4, we discuss new formalism to compute the partition functions of instantons whose moduli dynamics are not well-understood yet. We only focuss on the moduli space in the specific setting where all the massless moduli degrees are isolated by potentials from Coulomb VEVs and background fields. Throughout extending the classical instanton physics in the spirit of model building, we suggest a new formalism that provides the correct partition function. We test our new formalism by various other constructions from string theory. In chapter 5, we introduce another powerful method to compute the instanton partition function. Deforming the background geometry smoothly by blowing-up the origin, we derive the consistency condtion that partition functions of $\mathcal{N} = 2$ theories must satisfy. Analogy from the 4d, we also investigate similar consistency equations in 5d $\mathcal{N} = 1$ gauge theories. By enough number of consistency conditions, we compute the instanton partition functions just by knowing the perturbative ones.

# Chapter 2

# SUSY gauge theories in $d \geq 4$

In this chapter we review supersymmetric gauge theories with 8 supercharges in 4,5, and 6 dimensions. Firstly we introduce the Coulomb branch moduli space of vacua in 4d gauge theory, its effective dynamics and the role of instantons there. Then we consider consistent 5d and 6d gauge theories through Coulomb branch analysis in 5d, and anomaly cancelation conditions in 6d.

## 2.1 4d $\mathcal{N} = 2$ gauge theory and Coulomb branch

4d $\mathcal{N} = 2$ gauge theories have $SO(3,1)$ Lorentz symmetry, and $SU(2)_R \times U(1)_R$ R-symmetry. They consist of two kinds of $\mathcal{N} = 2$ multiplets; a vector multiplet $(A_\mu, \lambda_{\alpha A}, \phi)$ formed by gauge fileds, $SU(2)_R$ doublet gauginos, and one complex scalar field and a hypermultiplet $(q_A, \psi_\alpha)$ formed by $SU(2)_R$ doublet scalars and $SU(2)$ singlet fermions. Here $\alpha$ and $A$ subscripts denotes the doublet indices of 4d Weyl spinors and $SU(2)_R$ doublets. The Lagrangian is written by $\mathcal{N} = 1$

off-shell superfield formalsim as

$$\mathcal{L} = \int d^4\theta \, \mathrm{Tr}\left[\frac{\mathrm{Im}\,\tau}{4\pi}\Phi^\dagger e^{[V,\cdot]}\Phi + Q^\dagger\, e^V\, Q - \tilde{Q}\, e^{-V}\tilde{Q}^\dagger\right]$$
$$+ \int d^2\theta \frac{-i}{8\pi}\tau \, \mathrm{Tr}\left[W_\alpha W^\alpha + \tilde{Q}\Phi Q + m\tilde{Q}Q\right] + cc. \tag{2.1.1}$$

where $V, W_\alpha, \Phi, Q, \tilde{Q}$ denote superfields of the $\mathcal{N}=1$ vectormultiplet, its gauge invariant spinorial superfield form, the adjoint chiral multiplet in $\mathcal{N}=2$ vector multiplet, two chiral multiplets in the $\mathcal{N}=2$ hypermultiplet respectively. The coefficient $\tau$ is complexified coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$. The supersymmetric vacua are given by minimizing bosonic potentials conditions as following.

$$\frac{1}{g^2}[\Phi, \Phi^\dagger] + (QQ^\dagger - \tilde{Q}^\dagger\tilde{Q}) = 0,$$
$$\Phi Q + mQ = 0, \qquad \tilde{Q}\Phi + m\tilde{Q} = 0. \tag{2.1.2}$$

They allow two kinds of different vacua; (1) THe Higgs branch where $Q$ and $\tilde{Q}$ get non-trivial VEV only when $m=0$ and (2) the Coulomb branch where $\Phi$ get non-trivial VEV. In the Higgs branch, the rank of gauge group is reduced by Higgs mechanism. On the other hand in the Coulomb branch, the rank $r$ gauge group is broken to its subgroup $U(1)^r$ as vector multiplet scalar $\phi$'s cartan components get VEVs $\langle\phi^i\rangle \neq 0$.

The Coulomb branch effective action is completely determined by the prepotential $\mathcal{F}$ which is a holomorphic function of IR abelian vector superfield $\mathcal{A}$ [2, 102]. In the $\mathcal{N}=1$ superfield formalism it is written by

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\mathrm{Im}\left[\int d^4\theta \frac{\partial\mathcal{F}}{\partial\mathcal{A}}\bar{\mathcal{A}} + \frac{1}{2}\int d^2\theta \frac{\partial^2\mathcal{F}}{\partial\mathcal{A}^2}W_\alpha W^\alpha + cc\right]. \tag{2.1.3}$$

The classical Yang-Mills action contribute to the prepotential as

$$\mathcal{F}_{\text{cl}} = \frac{1}{2}\tau\,\mathcal{A}^2. \tag{2.1.4}$$

There is also 1-loop correction as

$$\mathcal{F}_{\text{pert}} = \frac{i}{2\pi}\mathcal{A}^2 \log \frac{\mathcal{A}^2}{\Lambda^2} \tag{2.1.5}$$

where $\Lambda$ is dynamically generated scale that is part of definition of the theory. Thanks to the supersymmetry there is no more perturbative corrections form higher loops. But there are non-perturbative corrections from instantons as

$$\mathcal{F}_{\text{inst}} = \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\mathcal{A}}\right)^{b_0 k} \mathcal{A}^2. \tag{2.1.6}$$

Here $k$ is the instanton number that is defined by a topological number in the Wick-rotated Euclidean $\mathbb{R}^4$

$$k = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} F \wedge F \tag{2.1.7}$$

and $b_0$ is the coefficient of the $\beta$-function of gauge coupling determined by Dynkin indices of representations of fermions that run the loop. In [2, 102], the prepotential was studied via a Seiberg-Witten curve that encodes the monodromy structure of the Coulomb branch moduli space. Later the non-perturbative prepotential was conjecturally computed by equivariant localization techniques on instanton moduli spaces in [3]. This conjecture was proven in [77,89] independently. We will introduce instanton moduli spaces and corresponding instanton partition functions in chapter 3. Some detailed examples appear in chapter 4.

## 2.2 5d $\mathcal{N} = 1$ gauge theories

5d $\mathcal{N} = 1$ theories have $SO(4,1)$ Lorentz symmetry and have 8 supercharges with $SU(2)_R$ symmetry. 5d $\mathcal{N} = 1$ QFTs consist of two kinds of supersymmetric multiplets; Vector multiplets $(A_\mu, \lambda, \phi)$ and hypermultiplets $(\varphi_\alpha, \psi)$. 5d $\mathcal{N} = 1$ gauge theories have Coulomb branch analogous to that of 4d $\mathcal{N} = 2$ theories. The Coulomb branch effective theory is described by the prepotential $\mathcal{F}$. The prepotential in 5d Coulomb branch is at most cubic to the $\mathcal{A}$ and its generic form is

$$\mathcal{F}(\phi) = \frac{1}{2}m_0\, h_{ij}\phi^i\phi^j + \frac{\kappa}{6}d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{12}\sum_{\boldsymbol{\alpha}\in\Delta}|\boldsymbol{\alpha}\cdot\boldsymbol{\phi}|^3 - \frac{1}{12}\sum_l\sum_{\boldsymbol{\omega}\in\boldsymbol{R}_l}|\boldsymbol{\omega}\cdot\boldsymbol{\phi}+m_l|^3 \tag{2.2.1}$$

where $h_{ij} = \mathrm{Tr}\,[T^iT^j]$ and $d_{ijk} = \frac{1}{2}\mathrm{Tr}\,[T^i\{T^j,T^k\}]$. Also $m_0$ is identified with $(\text{bare coupling})^{-2}$ and $\kappa$ is Chern-Simons number. $\kappa$ must be quantized properly so that the action in invariant under local gauge transformations. A vector multiplet is hodge dual to a rank 2 antisymmetric tensor multiplet which contains magnetic fields $B_{\mu\nu}$ and dual scalar $\phi_D^i = \frac{\partial\mathcal{F}}{\partial\phi^i}$. There are BPS states whose mass or tension is given by the scalar VEVs. The particles are electrically charged and the BPS mass of those particles with $n_e$ electric charge is

$$m_e = n_e^i\,\phi^i \tag{2.2.2}$$

while the strings are magnetically charged and their BPS saturated tensions for magnetic charge $n_m$ is

$$m_m = n_m^i\,\phi_D^i. \tag{2.2.3}$$

There are another BPS states related to the topological $U(1)_I$ symmetry whose current is defined as

$$j = \star(F\wedge F). \tag{2.2.4}$$

They are instantons and their masses are given by

$$m_I = k\,m_0 \tag{2.2.5}$$

where $k$ is the instanton number (2.1.7).

The topological $U(1)_I$ current is also 1-form and is carried by particles. As general prepotential contains cubic term in $\phi$, the instanton may couple with the gauge fields and get electrically charged. They are bound states of pure instantons and W-bosons, whose BPS saturated masses are given by sum of each BPS particle masses.

Since the prepotential is cubic in $\phi$, the Coulomb moduli metric $\tau_{ij}(\phi) = \frac{\partial^2 \mathcal{F}}{\partial\phi^i \partial\phi^j}$ is linear function of $\phi$. The theory is sensible only in the Coulomb moduli where its metric $\tau_{ij}(\phi) = \frac{\partial^2 \mathcal{F}}{\partial\phi^i \partial\phi^j} > 0$. When the negative linear part grows, the theory eventually hit the landau pole. However when the Coulomb moduli metric is postive definite in the entire Coulomb branch, there might be UV fixed point whose $m_0 = \frac{1}{g_0^2} = 0$. This necessary condition restricts the number of matters and their representations. And the actual existence of their UV fiexed points are checked by engineering 5d gauge theories from M-theory compactifed on Calabi-Yau 3-folds. Here we just introduce the results of [32].

- fundamental representation for $SU(N)$, $SO(N)$, $Sp(N)$, $G_2$, $F_4$, $E_6$, $E_7$
- antisymmetric representation for $SU(N)$, $Sp(N)$
- spinor representation for $SO(N)$ with $7 \leq N \leq 14$
- rank-3 antisymmetric representation for $Sp(3)$, $Sp(4)$, $SU(6)$, $SU(7)$
- symmetric representation for $SU(N)$.

In a last few years there has been various studies on extending above classifications. It was first pointed out that the previous arguement was totally perturbative and there are some region in the Coulomb branch where such perturbative argument doesn't work anymore. Those Coulomb branch are across the 'wall' of phase space where some non-perturbative degrees such as instantons or magnetic monopole string becomes massless or tensionless. The new classification suggested that there may exist a UV fixed point if the Coulomb moduli metric $\tau_{ij}$ is postive semi-definite inside those 'wall'. The extended classifications were done geometrically by M-theory compactified on Calabi-Yau 3-fold, or using 5-brane web-diagrams. Also more recently there have been studies of classification on 5d SCFTs flown from so-called 5d KK theories, that are 6d SCFTs compactified on a circle with some twist by discrete global symmetries around it. For more details, see [34, 48, 114, 117]

## 2.3 6d $\mathcal{N} = (1, 0)$ gauge theories

6d $\mathcal{N} = (1, 0)$ theories also have 8 suprcharges with $SU(2)_R$ symmetry. They have three kinds of SUSY multiplets; A vector multiplet $(A_\mu, \lambda)$, a hypermultiplet $(\varphi_\alpha, \psi)$ and a tensor multiplet $(B_{\mu\nu}, \chi, \Phi)$. There is no Coulomb branch in 6d $\mathcal{N} = (1, 0)$ theories since their vector multiplets do not contain scalars. Instead there are tensor branches where the tensor multiplet scalar get its non-zero VEV $\langle \Phi \rangle$. At tensor branch, the effective theory can be approximated to the 6d supersymmetric Yang-Mills theory, whose (gauge coupling)$^{-2}$ is proportional to the tensor VEV.

6d $\mathcal{N} = (1, 0)$ gauge theories are more restrictive than 5d $\mathcal{N} = 1$ or 4d $\mathcal{N} = 2$ theories because of guage anomlay. Unlike 4d $\mathcal{N} = 2$, gauginos and higgsinos in 6d $\mathcal{N} = (1, 0)$ multiplets have definite chiralities. Their 1-loop anomalies and gauge anomaly of effective action must cancel each other by Green-Schwarz mechanism. Furthermore, there are discrete anomalies when the gauge group $G$ has nontrivial $\pi_6(G)$ [37]. Such anomaly cancelation condition severely restricts the possible gauge group and the number of hypermultiplets.

6d $\mathcal{N}=(1,0)$ gauge theories shall also be defined with some UV-completion. A large class of 6d $\mathcal{N}=(1,0)$ SCFTs are classified geometrically. They were engineered from F-theory, which is the 12 dimensional theory that the axio-dilaton field in type IIB string theory is geometrically uplifted as the complex structure constant of the torus fibered over 10d. The 6d $\mathcal{N}=(1,0)$ theories can be constructed by compactifying F-theory on elliptically-fibered Calabi-Yau 3-folds (CY3). The base 4-manifold $B$ of the elliptically fibered CY3 may have nontrivial complex cycles. For a 6d SCFTs to exist, those complex cycles must be simultaneously shrunk so that the remaining theory is scale invariant. The 6d $\mathcal{N}=(1,0)$ SCFTs are classified by base 4-manifolds that meet the conditions [7, 11]. Here we list some SCFTs in [7].

| $G$ | $-$ | $-$ | $SU(3)$ | $SO(8)$ | $F_4$ | $E_6$ | $E_7$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $F$ | $E_8$ | $SO(5)_R$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| matters | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $\frac{1}{2}\mathbf{56}$ | $-$ |

| $G$ | $G_2 \times SU(2)$ | $G_2 \times Sp(1)$ | $SU(2) \times SO(7) \times SU(2)$ |
|---|---|---|---|
| matters | $\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$ | $\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$ | $\frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1})+\frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2})$ |

Note that there might be larger SCFTs which either can be reduced to one of above theories via Higgs mechanism, or can be constructed by gluing above theories obtained by comaptifying F-theory on an ellyptic fibered CY3 which has many mutually intersecting base 4-manifolds.

# Chapter 3

# Calculus of Classical Instantons

In this chapter we review the basic concepts of classical instantons, their moduli spaces and the methods to count them. We start from 4d Yang-Mills theory and will extend them to SUSY gauge theories in higher dimensions. Here 'classical' denotes for the theories with classical simple gauge groups coupled with matters in fundamental or antisymmetric representations. The moduli space of classical instantons can be constructed by so-called 'ADHM construction'.

## 3.1 4d Yang-Mills instantons

An instanton is defined on Euclidean $\mathbb{R}^4$. The 4d Yang-Mills action is

$$S = \int_{\mathbb{R}^4} -\frac{1}{4g^2} \operatorname{Tr} F_{mn} F_{mn} \tag{3.1.1}$$

where the field strength is defined as $F_{mn} = D_m A_n - D_n A_m = \partial_m A_n - \partial_n A_m + [A_m, A_n]$. The instanton is a solution of vacuum field equation

$$D_m F_{mn} = \partial_m F_{mn} + [A_m, F_{mn}] = 0 \tag{3.1.2}$$

that also satisfies the self-dual condition

$$F_{mn} = \frac{1}{2}\epsilon_{mnkl}F_{kl}. \tag{3.1.3}$$

It is characterized by instanton number $k$ (2.1.7). The instantons satifying self-dual condition has positive $k$, while the anti-instantons satisfying the anti self-dual condition has negative $k$. Its explicit solution was found firstly in pure $SU(2)$ Yang-Mills theory [4] with $k=1$ as

$$A_n = g^{-1}\frac{2\rho^2(x-X)_m\bar{\sigma}_{mn}}{(x-X)^2((x-X)^2+\rho^2)}. \tag{3.1.4}$$

Here the solution is parametrized by 8 parameters that are called collective coordinates. They consist of 4 dimensional position $X$, the instanton size $\rho$ and 3 orientations of $SU(2)$ global gauge orbit $g$. They are distinct solutions that can't be transformed to each other by local gauge transformations. The space of distinct solutions is called the instanton moduli space $\mathcal{M}_k$.

More generaly one can find the collective coordinates instanton solution as follows. Suppose there is a specific instanton solution $A_n$. Now we consider a small deformation along the collective coordinates $Z^i$

$$\delta A_n = Z_i\delta_i A_n = Z_i\left(\frac{\partial A_n}{\partial Z_i} + D_n\Omega_i\right) \tag{3.1.5}$$

where the last term comes from some local gauge tranformation. Since the deformed solution shall also be another solution, $\delta_i A_n$ should satisfy the linearized equation

$$D_m\delta_i A_n - D_n\delta_i A_m = \epsilon_{mnkl}D_k\delta_i A_l = 0. \tag{3.1.6}$$

It doesn't fully fix the collective coordinates since it is defined up to local gauge transformations. We must fix the gauge condition to fully fix the collective coordinates. One convenient way is to set the variation by collective coordinates 'orthogonal' to the arbitrary local gauge transformation, where the inner product is defined to be

$$\langle \delta A_n, \delta' A_n \rangle = -2 \int d^4x \, \mathrm{Tr} \, \delta A_n(x) \delta' A_n(x). \tag{3.1.7}$$

Then the orthogonality condition settles down to

$$\begin{aligned} \langle \delta_i A_n, D_n \Omega \rangle = -2 \int d^4x \, \mathrm{Tr} \, \delta_i A_n(x) \, D_n \Omega(x) = 0 \\ \rightarrow D_n \delta_i A_n = 0 \end{aligned} \tag{3.1.8}$$

after integration by parts in the second line. The natural metric of the moduli space is determined by the small deformation of the solution on collective coordinates,

$$g_{ij}(Z) = -\frac{2}{g_{YM}^2} \int d^4x \, \mathrm{Tr} \, [\delta_i A_m(Z) \cdot \delta_j A_m(Z)]. \tag{3.1.9}$$

For example, $SU(2)$ $k = 1$ instanton moduli space is $\mathbb{R}^4 \times \mathrm{Cone}[SU(2)]$ whose metric is given by

$$ds^2 = dX^i dX^i + d\rho^2 + \rho^2 d\Omega^2(SU(2)). \tag{3.1.10}$$

One instanton solution of classical gauge group $G$ can be constructed by embedding the $SU(2)$ instanton solution into global gauge orbit of $G$.

$$A_n = \mathcal{G} \begin{pmatrix} A_n^{SU(2)}|_{g=1} & 0 \\ 0 & 0 \end{pmatrix} \mathcal{G}^{-1} \tag{3.1.11}$$

where $\mathcal{G}$ is the gauge orientaition of $G$. Their moduli space is described by some cone of coset manifold

$$M_k = \mathbb{R}^4 \times \mathrm{Cone}[G/\tilde{G}] \tag{3.1.12}$$

with $\tilde{G}$ is the broken symmetry by $SU(2)$ embedding. For example if $G = SU(N)$, $\tilde{G} = S[U(N-2) \times U(1)]$. Note that it is singular at the point $\rho = 0$. It is the actual singularity that general instanton moduli space have. Such singularity is called small instanton singularity.

**ADHM Construction:** The arbitrary $k$ instanton solution of classical gauge group $G$ can be constructed by so-called the "ADHM" construction [1]. Here we shortly introduce the ADHM construction of $k$ instanton solution for $SU(N)$ Yang-Mills theory.

The ADHM formalism consist of 4 real $(k \times k)$ matrices $a_n = \frac{1}{\sqrt{2}}\bar{\sigma}_n^{\dot{\alpha}\alpha} a_{\alpha\dot{\alpha}}$, and two complex $(k \times N)$ matrices $q_{\dot{\alpha}}$ where $\alpha, \dot{\alpha} = 1, 2$. Here we use the notation of $\sigma_{n\alpha\dot{\alpha}} = (i\vec{\tau}, 1_{2\times 2})$ and $\bar{\sigma}_n^{\dot{\alpha}\alpha} = (-i\vec{\tau}, 1_{2\times 2})$ with Pauli matrices $\vec{\tau}$. These matrix degrees are called ADHM data. They are charged under rotation symmetry $SO(4) = SU(2)_l \times SU(2)_r$ where the $\alpha$ and $\dot{\alpha}$ indices denote for the doublet indices of each $SU(2)_{l,r}$. They are also charged under two kinds of gauge symmetries. The first one is the $SU(N)$ gauge symmetry of Yang-Mills theory, under which only the matrices $q_{\dot{\alpha}}$ are charged globally in the fundamental representation. There is additional auxiliary $U(k)$ symmetry which appears while constructing the instanton solution. The charges of ADHM data under symmetries are summarized in the following table.

| ADHM data | $U(k)$ | $SU(N)$ | $SU(2)_l$ | $SU(2)_r$ |
|---|---|---|---|---|
| $a_{\alpha\dot{\alpha}}$ | **Adj** | **1** | **2** | **2** |
| $q_{\dot{\alpha}}$ | **k** | $\overline{\mathbf{N}}$ | **1** | **2** |

However not all of them are independent collective coordinates and they satisfy the $(k \times k)$ matrix conditions called ADHM constraints

$$\vec{\tau}^{\dot{\alpha}}_{\dot{\beta}}(q_{\dot{\alpha}} q^{\dagger\dot{\beta}} + [a_{\alpha\dot{\alpha}}, a^{\dagger\alpha\dot{\beta}}]) = 0 \tag{3.1.13}$$

where $\vec{\tau}^{\dot{\alpha}}_{\dot{\beta}}$ are 3 Pauli matrices. Finally the auxiliary $U(k)$ must be moded out. Then the total number of degrees of freedom is

$$4k^2 + 4kN - 3k^2 - k^2 = 4kN. \tag{3.1.14}$$

The explicit instanton solution is constructed with ADHM data in the following manner. First we define the $(2k \times (N+2k))$ matrix

$$\Delta = \begin{pmatrix} q_{\dot{1}} & \\ & a_{\alpha\dot{\alpha}} + \sigma_{n\alpha\dot{\alpha}}(x-X)_n \\ q_{\dot{2}} & \end{pmatrix}. \tag{3.1.15}$$

Since $\Delta$ can have at maximal rank $2k$, its null-space is $N$ dimensional. The basis vector of that null-space can be written as $((N+2k) \times N)$ matrix $v$ satisfying

$$\Delta\, v(x) = 0 \tag{3.1.16}$$

that is normalized by

$$v^\dagger v = 1_{N\times N}. \tag{3.1.17}$$

Then the explicit instanton solution is written by

$$A_n = v(x)^\dagger \partial_n\, v(x). \tag{3.1.18}$$

The $SO(N)$ instanton solutions can also be constructed by regarding $SO(N)$ as $U(N)$ imposed by reality condition, as well as the $Sp(N)$ is regarded as $U(2N)$ emposed by pseudo-reality conditions. By (pseudo-)reality condition the auxiliary $U(k)$ gauge group of the solution is changed by $Sp(k)$ for $SO(N)$ instantons and $O(k)$ for $Sp(N)$ instantons. In such cases, the $a_{\alpha\dot{\alpha}}$ matrices are in the antisymmtric representation of $Sp(k)$ and symmetric representation of $O(k)$ for $SO(N)$ and $Sp(N)$ instantons respectively.

When the gauge theory has $\mathcal{N} = 2$ supersymmetry, the instanton configuration preserves only half of the SUSY by the SUSY invariance condition

$$\begin{aligned}
\delta\lambda^A &\sim \sigma^{mn} F_{mn} \xi^A = 0, \\
\delta\bar{\lambda}_A &\sim \bar{\sigma}^{mn} \bar{\xi}_A F_{mn} = 0
\end{aligned} \tag{3.1.19}$$

for chiral and antichiral gauginos. Since $\bar{\sigma}^{mn} F_{mn}$, only supersymmetry generated $\bar{\xi}_A$ is preserved. The broken supercharge generates fermi zero-modes of gaugino fields which correspond to the supersymmetric partner of bosonic zero-modes of gauge fields. In the end, they form kind of SUSY multiplet between bosonic and fermionic zero-modes.

## 3.2 Instantons in higher dimensions

Let us first review the instanton in 6d $\mathcal{N} = (1, 0)$ theories. Its bosonic part of effective action of the vector and tensor multiplets is written by

$$S_{\text{eff}} = \int \frac{1}{2} d\Phi \wedge \star d\Phi + \frac{1}{2} H \wedge \star H + \sqrt{c} \left( -\Phi \, \text{Tr} \left( F \wedge \star F \right) + B \wedge \text{Tr} \left( F \wedge F \right) \right) \tag{3.2.1}$$

where $H$ is the self-dual 3-form field strength of $B$. The action is effective in two different sense. First it is IR effective action where UV-completion is necessary for the theory to be well-defined. Secondly, it is not truly quantum mechanical action. The kinetic term of the $B$ field trivially vanishes as its field strenght $H$ is self-dual. One should take above Lagrangian as classical one from which the equations of motion are deriven by variation, and then the self-dual condition is given by hand.

The field equation of $B$ is

$$d \star H = \sqrt{c} \, \text{Tr}(F \wedge F). \tag{3.2.2}$$

It indicates that an instanton string solution(also called as "self-dual string") that extend along $\mathbb{R}^{1,1}$ has self-dual gauge fields in transverse $\mathbb{R}^4$ and sources the $H$ field. Its tension is proportional to the tensor VEV $\langle\Phi\rangle$. The moduli dynamics of instanton strings is described by two dimensional field theories defined on their worldsheet. It has 2d $\mathcal{N} = (0, 4)$ supersymmetry, which has 4 real supercharges preserved by instantons among 8 supercharges in 6d. Indeed, the worldsheet theory is exactly analogous to the ADHM formalism we reviewed in the previous section. The worldsheet 2d $\mathcal{N} = (0, 4)$ theory consist of 3 kinds of SUSY multiplets; a vector multiplet $(A_\mu, \lambda^A_{-\dot{\alpha}})$, a fundamental hypermultiplet $(q_{\dot{\alpha}}, \psi^A_+)$, an adjoint hypermultiplt $(a_{\alpha\dot{\alpha}}, \chi^A_{+\alpha})$. Its Lagrangian is written by standard kinetic terms, and complex D-term potentials that are minimized when the VEVs of scalar fields $q_{\dot{\alpha}}$, $a_{\alpha\dot{\alpha}}$ satisfy the ADHM constraints (3.1.13). Through the D-term potentials it realizes the instanton solution as classical moduli space.

Instanton particles in 5d $\mathcal{N} = 1$ gauge theories can be understood via 6d self-dual strings compactified on $S^1$. Its worldline quantum mechanics is exactly the one obtained by reducing above 2d theory on the worldsheet of a 6d self-dual strings on $S^1$. The quantum mechanics consists of similar multiplets with that of 2d theories while the vector multiplet now consist of one temporal gauge filed, gauginos and one real scalar $\varphi$ that corresponds to $A_1$ in 2d gauge theory. Since the 1d/2d gauge theories on worldline/worldsheet of 5d/6d instantons are extension of ADHM matrix model in 4d, they are often collectively called "ADHM formalism". Also as these theories flow to non-linear sigma models that have instanton moduli space as target space in IR, they are also called gauged linear sigma models(or GLSM). We exhibit more details of D-term potentials and corresponding moduli spaces in chapter 4.

One convenient way to construct the ADHM theory is to embed the gauge theory in string theories. When a gauge theory is realized by the low energy effective theory on the worldvolume of Dp-branes, it is known that a D(p-4) brane on them forms an instanton configuration of that gauge theory. Then the corresponding instanton dynamics is described by worldvolume theory of D(p-4) brane. At there instanton zero-modes are realized by open string modes between D-branes. By the string engineering, one can also construct the instanton dynamics including matters in fundamental, rank 2 (anti)symmetric representations. For more details, see [3, 24]. Let us comment about the limit of ADHM formalism. Unlike the classical gauge groups like $SU(N)$, $SO(N)$ and $Sp(N)$ are realized by stacks of Dp-branes and some orientifolds, exceptional gauge group appear only through some symmetry enhancements. Thus the manifest realization of exceptiontal gauge group is generically difficult and thus their ADHM formalism is unknown. Also since the open string has two ends, it is difficult to realize a gauge theory with matters in other representations via open strings.

## 3.3 Instanton partition function

In this section we shortly introduce how to compute instanton partition function (also referred as Nekrasov partition function). It is defined in the Coulomb branch on the so-called $\Omega$-background where chemical potentials of spacetime isometry and R-symmetry $SU(2)_l \times \mathrm{Diag}[SU(2)_r \times SU(2)_R] \subset SO(4) \times SU(2)_R$ is turned on. On such background every instanton zero-modes get massive by Coulomb VEVs, hypermultiplet masses and $\Omega$-background parameters $\epsilon_\pm$. Then the $k$ instanton partition function is defined by matrix integral over collective coordinates of $k$ instanton moduli space as

$$Z_k = \oint_{\tilde{M}_k} 1 = \int [\mathcal{D}X]\, e^{S_{\text{eff}}(X)} \tag{3.3.1}$$

where $X$ denotes every collective coordinates and $S_{\text{eff}}$ is the effective action of instanton zero-modes generated by background fields. By the localization, the matrix integral settles down to a rank $k$ contour integral of rational function

$$Z_k = \frac{1}{k!} \oint d^k\phi \frac{\prod_{\text{fermions}}(\vec{\phi}\cdot\vec{\Pi} + m_l \cdot F_l)}{\prod_{\text{bosons}}(\vec{\phi}\cdot\vec{\Pi} + m_l \cdot F_l)} \tag{3.3.2}$$

where denominators/numerators corresponds to masses of each bosonic/fermionic zero-modes. Here $\vec{\Pi}$ is charges of auxiliary gauge $\tilde{G}$ appears in ADHM formalism, $m_l$ denotes every background fields such as Coulomb VEVs $a_i$, Omega-background parameters $\epsilon_\pm$, hypermultiplet masses while $F_l$ denotes every charges conjugate to background fields $m_l$ such as $U(1)^r$ gauge charges, Cartan charges of $SU(2)_l$ and Diag$[SU(2)_r \times SU(2)_R]$, and other flavor charges. Since we turned on every chemical potentials of R-symmetry, it manifestly preserves only one supercharge that is neutral to Diag$[SU(2)_r \times SU(2)_R]$.

The contour was originally conjectured by Nekrasov for simple cases, and later was revealed to collect "Jeffrey-Kirwan" residues by studying elliptic genera of 2d $\mathcal{N} = (0,4)$ gauge theories [28, 29] and Witten indices of 1d quantum mechanics obtained by circle reduction of them [22, 26, 27], which are natural extensions of matrix integral formalism (3.3.1) to ADHM formalism for higher dimensional instantons on $\mathbb{R}^4 \times S^1$ and $\mathbb{R}^4 \times T^2$.

From an instanton partition function, the non-perturbative contribution to the Seiberg-Witten prepotential could be obtained by

$$\mathcal{F} = \lim_{\epsilon_{1,2}\to 0} \epsilon_1\epsilon_2\, Z_{\text{inst}}. \tag{3.3.3}$$

# Chapter 4

# 6d strings and exceptional instantons

In this chapter, we introduce a new ADHM-like formalism of $SO(N)$ gauge theories couple to hypermultiplets in spinor representations, and $G_2$ gauge theories with hypermultiplet in $\mathbf{7}$ representation. The new formalism is tested by other realizations using string theory embedding. We also apply our new formalism to compute elliptic genera of self-dual strings of non-Higgsable clusters in 6d $\mathcal{N} = (1,0)$ SCFTs.

## 4.1 Exceptional instanton partition functions

Our proposal is based on the following ideas: (1) We are interested in the Coulomb phase partition functions of exceptional instantons, not in the symmetric phase. (2) In the Coulomb phase, the instanton moduli space is lifted by massive parameters, to saddle points lying within the moduli space of instantons with classical subgroups. (3) Thus we only seek for a formalism to

| $G_r$ | $H_r$ | branching rules |
|---|---|---|
| $G_2$ | $SU(3)$ | $\mathbf{14} \to \mathbf{8} \oplus \mathbf{3} \oplus \overline{\mathbf{3}}$ , $\mathbf{7} \to \mathbf{3} \oplus \overline{\mathbf{3}} + \mathbf{1}$ |
| $F_4$ | $SO(9)$ | $\mathbf{52} \to \mathbf{36} \oplus \mathbf{16}$ , $\mathbf{26} \to \mathbf{1} \oplus \mathbf{9} \oplus \mathbf{16}$ |
| $E_7$ | $SU(8)$ | $\mathbf{133} \to \mathbf{63} \oplus \mathbf{70}$ , $\mathbf{56} \to \mathbf{28} \oplus \overline{\mathbf{28}}$ |
| $E_8$ | $SU(9)$ | $\mathbf{248} \to \mathbf{80} \oplus \mathbf{84} \oplus \overline{\mathbf{84}}$ |
| $E_8$ | $SO(16)$ | $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$ |
| $SO(7)$ | $SU(4)$ | $\mathbf{21} \to \mathbf{15} \oplus \mathbf{6}$ , $\mathbf{8} \to \mathbf{4} \oplus \overline{\mathbf{4}}$ |

Table 4.1: Possible choices of $H_r$ for various $G_r$, when $H_r$ is a simple group

study the massive fluctuations around the last saddle points, accomplished by extending ADHM formalisms for classical instantons. We elaborate on these ideas in some detail.

**Coulomb phase:** We are interested in the gauge theory in the Coulomb branch. Suppose that the gauge group $G_r$ has rank $r$. We turn on nonzero VEV $v$ of the scalar in the vector multiplet, which breaks $G_r$ to $U(1)^r$. In 6d, vector multiplet does not contain scalars. In this case, we consider the theory compactified on circle, with nonzero holonomy playing the role of Coulomb VEV. In the symmetric phase, instantons develop a moduli space, part of which being gauge orientations and instanton sizes. In the Coulomb phase, there appears nonzero potential on the instanton moduli space, proportional to $v^2$. This potential lifts the size and orientation 0-modes. There are extra $4k$ position moduli of $k$ instantons on $\mathbb{R}^4$, which will also be lifted in the Omega background. The moduli space is then completely lifted to points. So we expect that it suffices to understand the quantum dynamics of instantons near these points.

**ADHM on a subspace:** The second idea is that one can use the ADHM formalism of instantons when $G_r$ is a classical group. In $d$ dimensional gauge theory, the ADHM formalism can be understood as a $d-4$ dimensional gauge theory living on the instanton solitons. For classical $G_r$, the low energy moduli space of $d-4$ dimensional gauge theory is the instanton moduli space, so one expects in IR to get non-linear sigma models on the instanton moduli space. When $G_r$ is exceptional, no such formalisms are known. However, it is often possible to find a classical subgroup $H_r \subset G_r$ of the given exceptional group $G_r$ with same rank. Then, we try to describe the (massive) quantum fluctuations around the saddle points by expanding the $H_r$ ADHM formalism, adding more $d-4$ dimensional fields. This is where we need educated guesses, in the spirit of model buildings. We want a subgroup $H_r$ with same rank as $G_r$, partly because we wish our formalism to see all $U(1)^r$ in the Coulomb phase. Possible $G_r$ and $H_r$ are given in Table 4.1, when $H_r$ is a simple group. To study 'exceptional matters' of $SO(7)$, we shall also consider $H = SU(4)$ for $G = SO(7)$.

For example, consider the case with $H_{r=N-1} = SU(N)$. The $SU(N)$ ADHM description of $k$ instantons has $U(k)$ gauge symmetry, and the following fields,

$$\begin{aligned} \text{chiral} \quad &: \quad (q,\psi) \in (\mathbf{k}, \overline{\mathbf{N}}) \ , \quad (\tilde{q},\tilde{\psi}) \in (\bar{\mathbf{k}}, \mathbf{N}) \ , \quad (a,\Psi), \ (\tilde{a},\tilde{\Psi}) \in (\mathbf{adj}, \mathbf{1}) \\ \text{vector} \sim \text{Fermi} \quad &: \quad (A_\mu, \lambda_0) \in (\mathbf{adj}, \mathbf{1}) \ , \quad (\lambda) \in (\mathbf{adj}, \mathbf{1}) \ . \end{aligned} \tag{4.1.1}$$

The fields are organized into 2d $\mathcal{N} = (0,2)$ supermultiplets, and we have shown the representations in $U(k) \times SU(N)$. Fields in a parenthesis denote bosonic/fermionic ones in a multiplet, while $(\lambda)$ denotes a Fermi multiplet. These fields combine to $\mathcal{N} = (0,4)$ vector multiplet and hypermultiplets. The instanton moduli space is obtained from the scalar fields, subject to the complex ADHM constraint and the D-term constraint (real ADHM constraint)

$$q\tilde{q} + [a,\tilde{a}] = 0 \ , \quad qq^\dagger - \tilde{q}^\dagger \tilde{q} + [a,a^\dagger] + [\tilde{a},\tilde{a}^\dagger] = 0 \ , \tag{4.1.2}$$

and after modding out by the $U(k)$ gauge orbit. More precisely, the non-linear sigma model on the instanton moduli space is obtained from the gauged linear sigma model at low energy. This part is the standard ADHM construction of $SU(N)$ instantons. Now we should add extra light fields, including more scalars to describe $G_r$ instantons' extra moduli. $d$ dimensional vector multiplet in $G_r$ decomposes in $H_r$ as

$$\mathbf{adj}(G) \to \mathbf{adj}(H) \bigoplus_i \mathbf{R}_i(H) \ , \tag{4.1.3}$$

where $\mathbf{R}_i(H)$ are suitable representations of $H_r$ in Table 4.1. Vector multiplet in $\mathbf{adj}(H)$ induces the standard instanton moduli, described in UV by the above ADHM description. Vector multiplets in $\mathbf{R}_i$ introduce further moduli, whose real dimension is $4kT(\mathbf{R}_i)$. $T(\mathbf{R})$ is the Dynkin index of $\mathbf{R}$. When $\mathbf{R}_i$ is a fundamental representation or rank 2 product representations, we managed to find the extra fields. We are technically motivated by the mathematical constructions of [24], but will simply present them as our 'ansatz' for the UV uplift of these zero modes. From Table 4.1, one finds that $\mathbf{R}_i$'s are product representations with ranks less than or equal to 2 only for $G_2 \supset SU(3)$ and $SO(7) \supset SU(4)$. For these, the adjoint representations of $G_r$ decompose as

$$\begin{aligned} SU(3) \subset G_2 \quad &: \quad \mathbf{14} \to \mathbf{8} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{3} \oplus \mathbf{anti}(\mathbf{3} \otimes \mathbf{3}) \\ SU(4) \subset SO(7) \quad &: \quad \mathbf{21} \to \mathbf{15} \oplus \mathbf{6} \ . \end{aligned} \tag{4.1.4}$$

We shall present extra chiral and Fermi multiplets with suitable interactions in the next subsections, which extends the moduli space in $\sum_i 4kT(\mathbf{R}_i)$ new directions.

**When it fails:** We made similar trials with other exceptional gauge groups and matters, which failed. It may be worthwhile to briefly report the reasons of failure. A typical reason is that the UV theory has extra branch of moduli space that does not belong to our instanton moduli space at low energy. Namely, apart from the exceptional instanton's moduli space, one sometimes has extra branch which cannot be lifted by supersymmetric potentials.

For instance, we tried to extend the ADHM construction of $H = SU(8), SU(9)$ to get those for $G = E_7, E_8$. In these cases, $\mathbf{R}_i$'s are product representations of rank 4 and 3, respectively. We made several trials to realize the extra moduli with right dimensions, especially without unwanted extra branches of moduli space which may spoil the instanton calculus. We however failed to get the precise descriptions, despite finding models which partly exhibit the right physics of instantons and instanton strings. See section 5 for more discussions.

We also suspect that some choice of $H_r \subset G_r$ may miss certain small instanton saddle points. To clearly understand this issue, we should find more examples than we have now.

There are simpler examples in which our new formalism fails. For instance, we consider our alternative $SO(7)$ ADHM (section 2.1) and try to add zero modes from matters in $\mathbf{7}$ (vector representation). Zero modes of $\mathbf{7}$ are well known in the standard $SO(N)$ ADHM, which form an $Sp(k)$ fundamental Fermi multiplet. In our $SU(4) \times U(k)$ formalism, $\mathbf{7}$ is regarded as $\mathbf{7} \to \mathbf{6} + \mathbf{1}$. We can ignore the singlet if the gauge orientation of instantons is along $SU(4)$. $\mathbf{6}$ is the rank 2 anti-symmetric representation. According to [24], and in D-brane engineerings, the ADHM fields induced by matters in bulk anti-symmetric representation include scalars in rank 2 symmetric representation of $U(k)$. This creates an extra branch of moduli space which is unphysical in the instanton calculus, but is present only in the UV uplifts. Even in ADHM models engineered by string theory, there are often such extra branches. In [22], the contributions from these branches are factored out, mainly guided by string theory. However, including this extra branch in our $SO(7)$ ADHM-like model, we find it difficult to properly identify and separate the extra contributions. Similarly, we cannot do an ADHM-like calculus for the 5d $G_2$ $\mathcal{N}=1^*$ theory.

Now we explain examples that turn out to work.

#### 4.1.1 $SO(7)$ instantons and matters in 8

The adjoint representation of $SO(7)$ decomposes in $SU(4)$ as $\mathbf{21}\to\mathbf{15}+\mathbf{6}$. We first seek for an alternative ADHM-like formalism of pure $SO(7)$ instantons, extending $SU(4)$ ADHM. We explain it as the quantum mechanics of instanton particles in 5d $\mathcal{N}=1$ Yang-Mills theory.

The quantum mechanics for $k$ $SU(4)$ instantons has $U(k)$ gauge symmetry. It has following fields: $\mathcal{N}=(0,4)$ $U(k)$ vector multiplet, consisting of 1d reduction of 2d gauge fields $A_\mu=(A_0,\varphi=A_1)$, and fermions $\lambda_0,\lambda$; hypermultiplets with bosonic fields $q_i,\tilde{q}^i$ in $(\mathbf{k},\bar{\mathbf{4}})+(\bar{\mathbf{k}},\mathbf{4})$, where $i=1,2,3,4$; hypermultiplets with bosonic fields $a,\tilde{a}$ in $(\mathbf{adj},\mathbf{1})$. In IR, one imposes

$$D\sim qq^\dagger-\tilde{q}^\dagger\tilde{q}+[a,a^\dagger]+[\tilde{a},\tilde{a}^\dagger]=0\ ,\quad J_\lambda\sim q\tilde{q}+[a,\tilde{a}]=0 \qquad (4.1.5)$$

by D-term or $J$-term potentials in the $\mathcal{N}=(0,2)$ language. See, [5, 46, 47] for the notations and reviews. These constraints and modding out by $U(k)$ gauge orbit eliminate $2k^2$ complex variables from $2k^2+8k$ components of $q,\tilde{q},a,\tilde{a}$. So one finds $8k$ complex moduli.

With extra vector multiplet fields in $\mathbf{6}$, there are extra bosonic zero modes. A vector multiplet in rank 2 antisymmetric representation of $SU(N)$ induces $2kT(\mathbf{anti}_2)=k(N-2)$ complex bosonic zero modes in $k$ instanton background. So we should add extra fields in UV and modify interactions, to get extra $2k$ complex bosonic modes at $N=4$. We find that the following extra fields, taking the forms of $\mathcal{N}=(0,2)$ chiral or Fermi multiplets, yield the right physics[1] (only

[1] Technically, we took the equivariant index of the so-called universal bundle [24], and made

bosonic fields shown for the chiral multiplets):

$$\begin{aligned} \text{chiral } \phi_i &: (\bar{\mathbf{k}}, \bar{\mathbf{4}})_{J=\frac{1}{2}} \\ \text{chiral } b, \tilde{b} &: (\overline{\mathbf{anti}}_2, \mathbf{1})_{J=\frac{1}{2}} \\ \text{Fermi } \hat{\lambda} &: (\mathbf{sym}_2, \mathbf{1})_{J=0} \\ \text{Fermi } \check{\lambda} &: (\mathbf{sym}_2, \mathbf{1})_{J=-1} \ . \end{aligned} \tag{4.1.6}$$

$\mathbf{sym}_2$, $\mathbf{anti}_2$ denote rank 2 (anti-)symmetric representations of $\mathbf{k}$, and the charge $J$ in the subscript will be explained shortly. We introduced extra $4k + 2 \cdot \frac{k^2-k}{2}$ complex bosonic fields. Using the extra Fermi fields $\hat{\lambda}$, $\check{\lambda}$, we introduce the following interactions. As noted in [5], the desired interactions should be non-holomorphic in the chiral multiplet fields, which is possible only with $\mathcal{N} = (0,1)$ SUSY. Therefore, we regard all these fields as $(0,1)$ superfields, as explained in [5], and turn on the following $\mathcal{N} = (0,1)$ superpotential,

$$J_{\hat{\lambda}}^{(0,1)} \sim (\phi_i q^{i\dagger})_S + (ba^\dagger + \tilde{b}\tilde{a}^\dagger)_S \ , \quad J_{\check{\lambda}}^{(0,1)} \sim (\phi_i \tilde{q}^i)_S + (\tilde{b}a - b\tilde{a})_S \ . \tag{4.1.7}$$

The subscripts $S$ denote symmetrization of the $k \times k$ matrices. We want (4.1.7) to be the only source of breaking $(0,2)$ SUSY to $(0,1)$ in the classical action. The D-term is given by

$$D \sim qq^\dagger - \tilde{q}^\dagger \tilde{q} - \phi^\dagger \phi + [a, a^\dagger] + [\tilde{a}, \tilde{a}^\dagger] - 2b^\dagger b - 2\tilde{b}^\dagger \tilde{b} \ . \tag{4.1.8}$$

Then, since $|J|^2$ appear in the bosonic potential for each $J$, one imposes at low energy extra $k^2 + k$ complex constraints from the new superpotentials. Collecting all, one finds

$$3 \cdot 4k + 2 \cdot k^2 + 2 \cdot \frac{k^2 - k}{2} - 2k^2 - 2 \cdot \frac{k^2 + k}{2} = 10k \tag{4.1.9}$$

---

an antisymmetrized product of two of them (for $\mathbf{6}$): this is the character for the fields we list in (4.1.6). Since we lack physical explanations of this procedure, we just present them as our ansatz for the UV theory.

| fields | $U(k)$ | $SU(4)$ | $U(1)_J$ | $SU(2)_l$ |
|---|---|---|---|---|
| $(q_i, \tilde{q}^i)$ | $(\mathbf{k}, \bar{\mathbf{k}})$ | $(\bar{\mathbf{4}}, \mathbf{4})$ | $\frac{1}{2}$ | $\mathbf{1}$ |
| $(a, \tilde{a})$ | $\mathbf{adj}$ | $\mathbf{1}$ | $\frac{1}{2}$ | $\mathbf{2}$ |
| $(\lambda_0, \lambda)$ | $\mathbf{adj}$ | $\mathbf{1}$ | $(0, -1)$ | $\mathbf{1}$ |
| $\phi_i$ | $\bar{\mathbf{k}}$ | $\bar{\mathbf{4}}$ | $\frac{1}{2}$ | $\mathbf{1}$ |
| $(b, \tilde{b})$ | $\overline{\mathbf{anti}_2}$ | $\mathbf{1}$ | $\frac{1}{2}$ | $\mathbf{2}$ |
| $(\hat{\lambda}, \check{\lambda})$ | $\mathbf{sym}_2$ | $\mathbf{1}$ | $(0, -1)$ | $\mathbf{1}$ |

Table 4.2: Charges/representations of fields in our $SO(7)$ ADHM-like model

complex bosonic zero modes. This agrees with the dimension $2kc_2$ of $SO(7)$ instanton moduli space, where $c_2(SO(2N+1)) = 2N-1$. The $SU(4)$ ADHM quantum mechanics with extra charged fields given by (4.1.6) is our proposed ADHM-like formalism for $SO(7)$ instantons.

We explain the symmetries of this model. All symmetries we explain below are compatible with the superpotentials. It first has $SU(4)$ symmetry. There is also a $U(1)_J$ symmetry, whose charges $J$ we already listed above when we introduced fields. There is also $SU(2)_l$, which rotates $a, \tilde{a}$ and also $b, \tilde{b}$ as doublets. The charges and representations are summarized in Table 4.2. This system has only $\mathcal{N} = (0,1)$ supersymmetry and $SU(4)$ global symmetry in UV. We assert that they enhance to $(0,4)$ SUSY and $SO(7)$ in IR, when we compute the Coulomb branch partition functions. $SO(7)$ enhancement will be visible as $SO(7)$ character expansions of the partition functions. In the context of SUSY enhancement, we claim that $U(1)_J$ enhances to $SU(2)_r \times SU(2)_R$, where $SO(4) = SU(2)_r \times SU(2)_l$ rotates the spatial $\mathbb{R}^4$ on which particles can move, and $SU(2)_R$ is the 5d R-symmetry. $J$ is identified as $J = \frac{J_r + J_R}{2}$, where $J_r, J_R$ are the Cartans of $SU(2)_r$, $SU(2)_R$. One Cartan is not visible in UV. The index of our models will agree with different computations whose settings manifestly preserve $(0,4)$ SUSY.

We study the moduli space, with and without 5d Coulomb VEV. For technical reasons, let us just consider the case with $k=1$. At $k=1$, the fields $b, \tilde{b}$ are absent, and $a, \tilde{a}$ are free fields for the center-of-mass motion. First consider the symmetric phase at $v=0$. One should solve the following equations:

$$|q_i|^2 - |\tilde{q}^i|^2 - |\phi_i|^2 = 0 \ , \quad q_i\tilde{q}^i = 0 \ , \quad \phi^{\dagger i} q_i = 0, \quad \tilde{q}^i\phi_i = 0 \ . \tag{4.1.10}$$

At $\phi_i = 0$, this is the equation for the $SU(4)$ instantons. This subspace is the cone over $SU(4)/U(2)$. Away from $\phi_i = 0$, although the dimension of the moduli space is same as the relative moduli space of an $SO(7)$ instanton, the two moduli spaces are different. The proper $SO(7)$ instanton moduli space is the cone over the $SO(7)/(SU(2)\times SO(3))$ coset, whose metric is given by the homogeneous metric. However, we find no $SO(7)$ isometry on our moduli space.

In the Coulomb branch and with the Omega deformation, the moduli space lifts to isolated points, on the $SU(4) \subset SO(7)$ instanton moduli space. To see this, we expand the studies of [25]. The Coulomb VEV $v_i$ ($i = 1, \cdots, 4$) satisfying $\sum_i v_i = 0$ couples to the 1d fields as follows. Let us denote by $\varphi \equiv A_1$ the scalar in the 1d vector multiplet. $v$ is a traceless diagonal matrix, with eigenvalues $v_i$. Nonzero $v$ changes the coupling to $\varphi$ as follows,

$$|\varphi q|^2 + |\tilde{q}\varphi|^2 + |\phi\varphi|^2 \to |\varphi q - qv|^2 + |v\tilde{q} - \tilde{q}\varphi|^2 + |v\phi + \phi\varphi|^2 \ . \tag{4.1.11}$$

This is because $\varphi$, $v$ are scalars in the 1d vector multiplet of $U(k)\times SU(4)$, where $v$ is a background field, and fields couple to them according to their representations in $U(k)\times SU(4)$. Note that the relative $-$ signs for $q, \tilde{q}$ appear because they are in the bifundamental representations $(\mathbf{k}, \bar{\mathbf{4}})$ or its conjugate, while relative $+$ sign for $\phi$ is because it is in $(\bar{\mathbf{k}}, \bar{\mathbf{4}})$. We set all complete square terms to zero at energies lower than 1d gauge coupling. One should also minimize the following D-term potential at $k=1$,

$$V \leftarrow \left(|q|^2 - |\tilde{q}|^2 - |\phi|^2 - \xi\right)^2 \ . \tag{4.1.12}$$

Here, since we have a $U(k)$ gauge theory, we have turned on a Fayet-Iliopoulos (FI) parameter, $\xi$, which we take to be positive $\xi > 0$ for convenience. $\xi$ can also be taken to be negative, without changing the Coulomb phase partition function, as we shall see below. However, physics is easier to interpret with $\xi > 0$. So we set (at $k=1$)

$$v_i q_i = \varphi q_i \ , \quad v_i \tilde{q}^i = \varphi \tilde{q}^i \ , \quad v_i \phi_i = -\varphi \phi_i \tag{4.1.13}$$

where $i = 1,2,3,4$ indices are not summed over, and

$$|q_i|^2 - |\tilde{q}^i|^2 - |\phi_i|^2 = \xi > 0 \ . \tag{4.1.14}$$

The equations (4.1.13) are eigenvector equations for the matrix $v$, whose eigenvalues are $\varphi$ for $q, \tilde{q}$ to be nonzero, and $-\varphi$ for $\phi$ to be nonzero. From (4.1.14), one should have $q_i \neq 0$, which means that $\varphi$ is set equal to one of the $v_i$'s. Then one can have nonzero $q_i$ at the saddle point, whose value is tuned to meet (4.1.14). At generic values of $v_i$'s, one should set $\phi_i = 0$, meaning that we are forced to stay in the $SU(4)$ instanton moduli space.[2] So in the Coulomb branch calculus, $\phi$ provides massive degrees of freedom living on the $SU(4)$ instanton moduli space.

The Witten index of the quantum mechanics preserving $(0,4)$ SUSY is defined by

$$Z_k(\epsilon_{1,2}, v_i) = \mathrm{Tr}_k \left[(-1)^F e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-v_i q_i} e^{-m_a F_a}\right] \ , \tag{4.1.15}$$

[2] At this stage, $\tilde{q}^i$ can also be nonzero by solving the same eigenvector equation as $q_i$. However, as shown in the appendix of [25], the eigenvector equations for $q_i$ and $\tilde{q}^i$ become different with nonzero Omega background parameter. Therefore, in the fully Omega-deformed background, only $q_i$ is nonzero.

where trace is over states in the $k$ instanton sector. $J_1, J_2$ are the two Cartans of $SO(4)$ which rotate the spatial $\mathbb{R}^4$, where they rotate mutually orthogonal $\mathbb{R}^2$ factors. They are related to $J_{l,r}$ by $J_r = \frac{J_1+J_2}{2}$, $J_l = \frac{J_1-J_2}{2}$. $J_R$ is the Cartan of $SU(2)_R$ coming from the 5d R-symmetry. Note that only the combination $J_r + J_R = 2J$ appears, so our UV model can fully detect them. $q_i$ are the $r$ electric charges in $U(1)^r \subset G_r$, which is $SO(7)$ here. $F_a$ denote other flavor symmetries, which is absent now but introduced for later purpose. The measures are chosen to commute with two Hermitian supercharges $Q^{A\dot{\alpha}} = Q^{+\dot{-}}, Q^{-\dot{+}}$. See, e.g. [22] for the notations. These two supercharges are mutually Hermitian conjugate, which we write as $Q, Q^\dagger$. They form a pair of fermionic oscillators, pairing a set of bosonic and fermionic states. Such a pair of states is not counted in the index, as their contributions cancel due to the factor $(-1)^F$. Such a Hilbert space interpretation will hold with as little as $(0,2)$ SUSY. In our UV $(0,1)$ system, we abstractly interpret the partition function as a SUSY path integral of the Euclidean QFT on $T^2$. 1 Hermitian SUSY in UV is enough to derive the formula for $Z_k$ available in the literatures. With IR SUSY enhancement, $Z_k$ acquires the interpretation of an index.

For gauge theories, this index can be evaluated by a residue sum [22, 26, 27] (see also [28,29]). The formula was discussed in the context of $(0,2)$ theories, but it applies with 1 Hermitian supercharge as well [5]. In our model, the contour integral takes the following form[3]:

$$\begin{aligned} Z_k &= \frac{1}{k!}\oint \prod_{I=1}^{k} \frac{d\phi_I}{2\pi i} \cdot \frac{\prod_{I\neq J} 2\sinh\frac{\phi_{IJ}}{2} \cdot \prod_{I,J} 2\sinh\frac{2\epsilon_+ - \phi_{IJ}}{2}}{\prod_{I=1}^{k}\prod_{i=1}^{4} 2\sinh\frac{\epsilon_+ \pm(\phi_I - v_i)}{2} \cdot \prod_{I,J} 2\sinh\frac{\epsilon_{1,2}+\phi_{IJ}}{2}} \\ &\quad \times \frac{\prod_{I\leq J}\left(2\sinh\frac{\phi_I+\phi_J}{2}\cdot 2\sinh\frac{\phi_I+\phi_J-2\epsilon_+}{2}\right)}{\prod_I\prod_i 2\sinh\frac{\epsilon_+-\phi_I-v_i}{2}\cdot\prod_{I<J}2\sinh\frac{\epsilon_{1,2}-\phi_I-\phi_J}{2}}\ . \end{aligned} \tag{4.1.16}$$

[3]The overall signs of $Z_k$ are fixed by requiring agreement with the index for the $Sp(k)$ ADHM theory [21].

Here, $\phi_{IJ} \equiv \phi_I - \phi_J$, and $2\sinh$ factors with repeated signs or subscripts (like $\pm$ or $\epsilon_{1,2}$) are all multiplied. The $SU(4)$ chemical potentials satisfy $\sum_{i=1}^{4} v_i = 0$. We also used $\epsilon_\pm \equiv \frac{\epsilon_1 \pm \epsilon_2}{2}$. The integrand on the first line comes from the $SU(4)$ ADHM fields $q, \tilde{q}, a, \tilde{a}$ and $U(k)$ vector multiplet fermions. The second line comes from the extra fields.

The integral can be performed as follows. The nonzero residue contributing to $Z_k$ is called the JK residue. To define this, one first picks up an auxiliary vector $\eta$ in the $k$ dimensional charge space ('conjugate' to the integral variables $\phi_I$). Possible poles in the integrand are given by hyperplanes of the form $\rho_\alpha \cdot \phi + \cdots = 0$, where the expression on the left hand side comes from the argument of the sinh factors $2\sinh \frac{\rho_\alpha \cdot \phi + \cdots}{2}$ in the denominator of (4.1.16). One can in general pick $d(\geq k)$ charge vectors $\rho_\alpha$, $\alpha = 1, \cdots, d$ and hyperplanes to specify a pole. In our systems, all relevant poles satisfy $d = k$. With chosen $\eta$, JK-Res may be nonzero only if $\eta$ is spanned by the $k$ charge vectors $\rho_1, \rho_2, \cdots, \rho_k$ with positive coefficients. Here, the choice $\eta = (1, \cdots, 1)$ simplifies the evaluation [22]. Since the charges appearing in the denominator of the second line are all negative in (4.1.16), one can show (combined with the fact that charges on the first line take the form of $e_I$ or $e_I - e_J$) that JK-Res should always be zero by definition if one of the charges from the second line are chosen in $\rho_\alpha$. This implies that the poles with nonzero residues are always chosen from the first line only, which are already classified in [3, 22, 30, 31]. The pole locations for $\phi_I$ are classified by the colored Young diagrams with $k$ boxes, meaning a collection of 4 Young diagrams $Y = (Y_1, \cdots, Y_4)$ whose box numbers sum to $k$. Let us denote by $s = (m, n)$ the box of a Young diagram $Y_i$, which is the box on the $m$'th row and $n$'th column of $Y_i$. $s$ running over possible $k$ boxes replaces $I = 1, \cdots, k$ index of $\phi_I$. We specify the pole location associated with $Y$ as $\phi(s)$. The result is [3,22,30,31]

$$\phi(s) = v_i - \epsilon_+ - (n-1)\epsilon_1 - (m-1)\epsilon_2 \ , \quad s = (m,n) \in Y_i \quad (i = 1, \cdots, 4) \ . \quad (4.1.17)$$

(This corrects a typo in [22], exchanging $m \leftrightarrow n$.) Had there been only the first line in (4.1.16), the residues were computed in [22,30,31]. Plugging in $\phi(s)$ into the second line of (4.1.16), one obtains an extra factor for each residue. The residue sum is given by

$$\begin{aligned} Z_k &= \sum_{\vec{Y};|\vec{Y}|=k} \prod_{i=1}^{4} \prod_{s \in Y_i} \frac{2\sinh(\phi(s)) \cdot 2\sinh(\phi(s) - \epsilon_+)}{\prod_{j=1}^{4} 2\sinh\frac{E_{ij}(s)}{2} \cdot 2\sinh\frac{E_{ij}(s)-2\epsilon_+}{2} \cdot 2\sinh\frac{\epsilon_+ - \phi(s) - v_j}{2}} \\ &\quad \times \prod_{i \le j}^{4} \prod_{s_{i,j} \in Y_{i,j}; s_i < s_j} \frac{2\sinh\frac{\phi(s_i)+\phi(s_j)}{2} \cdot 2\sinh\frac{\phi(s_i)+\phi(s_j)-2\epsilon_+}{2}}{2\sinh\frac{\epsilon_{1,2}-\phi(s_i)-\phi(s_j)}{2}} \end{aligned} \quad (4.1.18)$$

where

$$E_{ij}(s) = v_i - v_j - \epsilon_1 h_i(s) + \epsilon_2 (v_j(s) + 1) \ . \quad (4.1.19)$$

Here and below, $s_i < s_j$ means $(i < j)$ or $(i = j$ and $m_i < m_j)$ or $(i = j$ and $m_i = m_j$ and $n_i < n_j)$. $h_i(s)$ denotes the distance from $s$ to the right end of the diagram $Y_i$ by moving right. $v_j(s)$ denotes the distance from $s$ to the bottom of the diagram $Y_j$ by moving down. See, e.g. [25]. (4.1.18) is our proposal for the partition function of $k$ $SO(7)$ instantons. This is quite novel for the following reason. $SO(7)$ instantons have standard ADHM formulation, using $Sp(k)$ gauge theories for $k$ instantons. The pole classification is unknown for the $Sp(k)$ index. On the other hand, (4.1.18) is an explicit formula.

Before adding matters in **8**, we first check that (4.1.18) is indeed the correct $SO(7)$ instanton partition function. We checked the equivalence of (4.1.16), or (4.1.18), and the index of $Sp(k)$ ADHM gauge theory [21,22], up to $k \le 3$ (turning off all chemical potentials except $\epsilon_+$ at $k = 3$). Here we explain the case with $k = 1$ in detail, which is already nontrivial. For the purpose of illustration, we directly start from the contour integral. At $k=1$, one finds

$$\left(2\sinh\frac{\epsilon_{1,2}}{2}\right)Z_1=\oint d\phi\frac{2\sinh\epsilon_+}{\prod_{i=1}^4 2\sinh\frac{\epsilon_+\pm(\phi-v_i)}{2}}\cdot\frac{2\sinh\phi\cdot 2\sinh(\phi-\epsilon_+)}{\prod_{i=1}^4 2\sinh\frac{\epsilon_+-\phi-v_i}{2}} \tag{4.1.20}$$

from our model. Taking the residues at $\phi=v_i-\epsilon$, for $\eta>0$, one finds

$$\left(2\sinh\frac{\epsilon_{1,2}}{2}\right)Z_1=\sum_{i=1}^4\frac{1}{\prod_{j(\neq i)}2\sinh\frac{v_{ij}}{2}\cdot 2\sinh\frac{2\epsilon_+-v_{ij}}{2}}\cdot\frac{2\sinh(2\epsilon_+-v_i)}{\prod_{j(\neq i)}2\sinh\frac{2\epsilon_+-v_i-v_j}{2}}\,. \tag{4.1.21}$$

This is a special case of (4.1.18). To check this result is correct, we study the $SO(7)$ single instanton partition function obtained from the standard $Sp(1)$ ADHM formalism [21, 22]

$$\left(2\sinh\frac{\epsilon_{1,2}}{2}\right)Z_1^{\rm standard}=\frac{1}{2}\oint d\phi\frac{2\sinh\epsilon_+\cdot 2\sinh(\epsilon_+\pm\phi)\cdot 2\sinh(\pm\phi)}{\prod_{a=1}^3 2\sinh\frac{\epsilon_+\pm\phi\pm u_a}{2}\cdot 2\sinh\frac{\epsilon_+\pm\phi}{2}}\,. \tag{4.1.22}$$

Residues are taken at $\phi=\pm u_a-\epsilon_+$ and $-\epsilon_+$ for $\eta>0$, but the last residue is 0. $u_a$ and $v_i$ are related by

$$v_1=\frac{u_1+u_2+u_3}{2}\,,\ v_2=\frac{u_1-u_2-u_3}{2}\,,\ v_3=\frac{-u_1+u_2-u_3}{2}\,,\ v_4=\frac{-u_1-u_2+u_3}{2}\,. \tag{4.1.23}$$

The residue sum is given by

$$\left(2\sinh\frac{\epsilon_{1,2}}{2}\right)Z_1^{\rm standard}=\frac{1}{2}\sum_{a=1}^3\sum_{s=\pm}\frac{2\cosh\frac{su_a}{2}\cdot 2\cosh\frac{2\epsilon_+-su_a}{2}\cdot 2\sinh(\pm(su_a-\epsilon_+))}{2\sinh(\frac{\epsilon_+\pm(-\epsilon_++2su_a)}{2})\prod_{b(\neq a)}2\sinh\frac{\epsilon_+\pm(-\epsilon_+su_a\pm u_b)}{2}}\,. \tag{4.1.24}$$

Despite very different looks, one can show (say, by using computer) that

$$Z_1(v_i)=Z_1^{\rm standard}(u_a) \tag{4.1.25}$$

after the identification (4.1.23). This identity and similar ones at higher $k$'s imply that $Z_k$ exhibits $SU(4)\to SO(7)$ enhancement, since $Z_k^{\rm standard}$ has manifest $SO(7)$ Weyl symmetry.

Now we discuss the inclusion of ADHM fields coming from the hypermultiplet matters in $\mathbf{8}$. We continue to study the instanton particles of 5d SYM. $\mathbf{8}$ decomposes in $SU(4)$ as

$$\mathbf{8} \to \mathbf{4} + \bar{\mathbf{4}} \ . \tag{4.1.26}$$

In the original ADHM formalism of $SO(7)$ instantons, it is unclear how to UV-uplift the fermion zero modes caused by these hypermultiplets in the instanton background. One may even feel it impossible, since the standard $SO(7)$ ADHM cannot see $4\pi$ rotations in $Spin(7)$. However, viewing it as $SU(4)$ instantons with certain extensions, each hypermultiplet in $\mathbf{4}$ (or $\bar{\mathbf{4}}$) induces a Fermi multiplet which is fundamental (or anti-fundamental) in $U(k)$. So in our new description, we naturally guess that the effect of $n_{\mathbf{8}}$ hypermultiplets is adding $n_{\mathbf{8}}$ pairs of Fermi multiplets of the following form:

$$\Psi_a, \tilde{\Psi}_a \ : \ (\mathbf{k}, \mathbf{1}) + (\bar{\mathbf{k}}, \mathbf{1}) \qquad (a = 1, \cdots, n_{\mathbf{8}}) \ . \tag{4.1.27}$$

It has been known [32] that the 5d $SO(7)$ SYM has a UV completion to a 5d SCFT for $n_{\mathbf{8}} \leq 4$. Recently, it was discussed that 5d SCFTs can exist till $n_{\mathbf{8}} \leq 6$ [33]. See also [34]. Our construction provides good descriptions of instantons for $n_{\mathbf{8}} \leq 4$. It will be easiest to explain this point after we discuss the index below. The flavor symmetry for $\Psi_a, \tilde{\Psi}_a$ may naively appear to be $U(2n_{\mathbf{8}})$. This is because we do not have any superpotential for these Fermi fields. They interact with other fields through gauge coupling only, so that one can rotate $\Psi_a, \tilde{\Psi}_a^\dagger$ with $U(2n_{\mathbf{8}})$. However, these fermions can couple to 5d background bulk fields, including the hypermultiplet fields in $\mathbf{8}$. (Even in ADHM models based on D-brane engineerings, it sometimes happens that the soliton quantum mechanics is ignorant on the bulk symmetry, in a similar manner.) These couplings will only preserve $U(n_{\mathbf{8}}) \subset Sp(n_{\mathbf{8}})$. See the beginning of the next subsection for this coupling to the bulk fields.

Adding these fermions, our ADHM-like description can be easily generalized. Namely, the extra Fermi fields are given standard kinetic term, whose derivatives are covariantized with 1d $U(k)$ vector multiplet fields. Its Witten index $Z_k^{n_{\mathbf{8}}}(\epsilon_{1,2}, v_i, m_a)$ with $a = 1, \cdots, n_{\mathbf{8}}$ is defined with extra factors $e^{-m_a F_a}$ inserted in its definition, where $F_a$ are the Cartans of $Sp(n_{\mathbf{8}})$. The contour integral expression for the Witten index takes the form of (4.1.16), with the following extra integrand multiplied for the new Fermi fields:

$$\prod_{a=1}^{n_{\mathbf{8}}} \prod_{I=1}^{k} 2 \sinh \frac{m_a + \phi_I}{2} \cdot 2 \sinh \frac{m_a - \phi_I}{2} \ . \tag{4.1.28}$$

The extra factor (4.1.28) does not create new poles at finite $\phi$, but may create new poles at infinity $\phi_I \to \pm\infty$. We first discuss the last possibility.

Here, first note that $\phi_I$ originate from the eigenvalues of the 1d $U(k)$ vector multiplet fields, $\phi \equiv \varphi + iA_\tau$, where $A_\tau$ is the vector potential on the Euclidean time. The contour integrand $Z_{\text{1-loop}}$ comes from 1-loop path integral of 1d fields in the background of constant $\phi_I$. So $V(\phi) \sim -\log Z_{\text{1-loop}}$ is the 1-loop potential energy for $\phi_I$. Before multiplying (4.1.28), the integrand of (4.1.16) converges to zero at $|\phi_I| \to \infty$ for any $I$, since there are more bosonic fields than fermionic fields. More concretely, consider the case with $k = 1$. One obtains $Z_{\text{1-loop}}^{n_{\mathbf{8}}=0} \sim e^{-4|\phi|}$, implying that the linear potential $V(\phi) = 4|\phi|$ confines the eigenvalues to $\phi = 0$. In other words, although $\phi$ classically develops a continuum to $\phi \to \pm\infty$, 1-loop effect lifts this continuum by an attractive force. In ADHM models with brane engineering, this can be visualized as the instantons being attracted to the locations of 5d SCFTs [22]. The $U(k)$ vector multiplet fields are clearly extra degrees of freedom that enter while making a UV completion of the nonlinear sigma model. If there is a continuum created by $\phi$, this represents states that do not belong to 5d QFT. The confinement from $V(\phi) = 4|\phi|$ signals that such obvious extra states may not be present in the quantum system.

Now we extend these studies to $n_{\mathbf{8}} > 0$. At $k = 1$, one obtains $V(\phi) = (4 - n_{\mathbf{8}})|\phi|$. So at $n_{\mathbf{8}} \leq 3$, the quantum potential still confines the instanton. At $n_{\mathbf{8}} = 4$, $\phi$ generates a flat direction. This branch has extra states which is an artifact of UV completion, not belonging to the 5d QFT Hilbert space. So strictly speaking, $n_{\mathbf{8}} \leq 3$ is the bound in which our ADHM-like model is reliable. Fortunately, there are well developed empirical ways of factoring out such extra states' contribution to the index. So we believe that our approach will be useful till $n_{\mathbf{8}} = 4$. At $n_{\mathbf{8}} \geq 5$, the quantum potential is repulsive, and it is not clear whether one can use this theory to study 5d QFT at all. (However, see [35] for some progress.) In the contour integral like (4.1.16) or its extension with (4.1.28), the absence of continuum means the absence of poles at infinity. This implies that the choice of $\eta$ in the JK-residue evaluation does not change the final result [22, 26]. This is the case for $n_{\mathbf{8}} \leq 3$.

For $n_{\mathbf{8}} \leq 3$, the pole classification that we explained earlier for pure $SO(7)$ instantons still holds, labeled by $SU(4)$ colored young diagrams. We only need to multiply the value of (4.1.28) at the pole to the residue. The resulting index is given by

$$\begin{aligned} Z_k &= \sum_{\vec{Y};|\vec{Y}|=k} \prod_{i=1}^{4} \prod_{s\in Y_i} \frac{2\sinh(\phi(s)) \cdot 2\sinh(\phi(s)-\epsilon_+)}{\prod_{j=1}^{4} 2\sinh\frac{E_{ij}(s)}{2} \cdot 2\sinh\frac{E_{ij}(s)-2\epsilon_+}{2} \cdot 2\sinh\frac{\epsilon_+-\phi(s)-v_j}{2}} \\ &\times \prod_{i\leq j}^{4} \prod_{s_{i,j}\in Y_{i,j};s_i<s_j} \frac{2\sinh\frac{\epsilon_+\pm(-\epsilon_+\phi(s_i)+\phi(s_j))}{2}}{2\sinh\frac{\epsilon_{1,2}-\phi(s_i)-\phi(s_j)}{2}} \cdot \prod_{i=1}^{4}\prod_{s\in Y_i}\prod_{a=1}^{n_{\mathbf{8}}} 2\sinh\frac{m_a\pm\phi(s)}{2}\,. \end{aligned} \tag{4.1.29}$$

The partition functions (4.1.29) will be tested in sections 3 and 4 at $n_{\mathbf{8}} = 1, 2$ using alternative descriptions, which include no guess works but are more elaborate in calculations. For instance, the indices at $k = 1$ divided by the center-of-mass factor $\hat{Z}_1 \equiv \left(2\sinh\frac{\epsilon_{1,2}}{2}\right) Z_1$ are given by

$$\hat{Z}_1^{n_{\mathbf{8}}=1} = \prod_{i<j} \frac{t^2}{(1-t^2 b_i^{\pm} b_j^{\pm})} \left[\chi_{\mathbf{2}}^{Sp(1)} f_0(v) + f_1(v)\right] \tag{4.1.30}$$

$$\hat{Z}_1^{n_{\mathbf{8}}=2} = \prod_{i<j} \frac{t^2}{(1-t^2 b_i^{\pm} b_j^{\pm})} \left[\chi_{\mathbf{5}}^{Sp(1)} f_0(v) + \chi_{\mathbf{4}}^{Sp(2)} f_1(v) + f_2(v)\right] \tag{4.1.31}$$

$$\hat{Z}_1^{n_{\mathbf{8}}=3} = \prod_{i<j} \frac{t^2}{(1-t^2 b_i^{\pm} b_j^{\pm})} \left[\chi_{\mathbf{14'}}^{Sp(3)} f_0(v) + \chi_{\mathbf{14}}^{Sp(3)} f_1(v) + \chi_{\mathbf{6}}^{Sp(3)} f_2(v) + f_3(v)\right] \tag{4.1.32}$$

where

$$\begin{aligned} f_0(v) &= \chi_{\mathbf{9}}^{SU(2)} + \chi_{\mathbf{7}}^{SU(2)}(\chi_{\mathbf{7}}^{SO(7)}+1) + \chi_{\mathbf{5}}^{SU(2)}(-\chi_{\mathbf{35}}^{SO(7)} + \chi_{\mathbf{7}}^{SO(7)} + 1) \\ &\quad + \chi_{\mathbf{3}}^{SU(2)}(-\chi_{\mathbf{35}}^{SO(7)} + \chi_{\mathbf{27}}^{SO(7)} + 1) + \chi_{\mathbf{105}}^{SO(7)} - \chi_{\mathbf{21}}^{SO(7)} + \chi_{\mathbf{7}}^{SO(7)} \\ f_1(v) &= -\chi_{\mathbf{8}}^{SU(2)}\chi_{\mathbf{8}}^{SO(7)} - \chi_{\mathbf{6}}^{SU(2)}\chi_{\mathbf{8}}^{SO(7)} + \chi_{\mathbf{4}}^{SU(2)}\chi_{\mathbf{112}}^{SO(7)} - \chi_{\mathbf{2}}^{SU(2)}\chi_{\mathbf{168}}^{SO(7)} \\ f_2(v) &= \chi_{\mathbf{7}}^{SU(2)}\chi_{\mathbf{35}}^{SO(7)} - \chi_{\mathbf{5}}^{SU(2)}\big(\chi_{\mathbf{7}}^{SO(7)} - \chi_{\mathbf{35}}^{SO(7)} + \chi_{\mathbf{105}}^{SO(7)}\big) - \chi_{\mathbf{3}}^{SU(2)}\big(\chi_{\mathbf{21}}^{SO(7)} + \chi_{\mathbf{27}}^{SO(7)} \\ &\quad - \chi_{\mathbf{35}}^{SO(7)} - \chi_{\mathbf{77}}^{SO(7)} + \chi_{\mathbf{168'}}^{SO(7)} + 1\big) - \chi_{\mathbf{7}}^{SO(7)} + \chi_{\mathbf{21}}^{SO(7)} + \chi_{\mathbf{27}}^{SO(7)} \\ &\quad - \chi_{\mathbf{105}}^{SO(7)} + \chi_{\mathbf{189}}^{SO(7)} + \chi_{\mathbf{330}}^{SO(7)} \\ f_3(v) &= -\chi_{\mathbf{112'}}^{SO(7)}\chi_{\mathbf{6}}^{SU(2)} + (\chi_{\mathbf{48}}^{SO(7)} - \chi_{\mathbf{112'}}^{SO(7)} + \chi_{\mathbf{512}}^{SO(7)})\chi_{\mathbf{4}}^{SU(2)} - (\chi_{\mathbf{112'}}^{SO(7)} + \chi_{\mathbf{448}}^{SO(7)})\chi_{\mathbf{2}}^{SU(2)} . \end{aligned} \tag{4.1.33}$$

Here $t \equiv e^{-\epsilon_+}$. $\chi_{\mathbf{R}}^{SU(2)}$ is the character of diag$[SU(2)_R \times SU(2)_r]$ in representation $\mathbf{R}$, in the convention $\chi_{\mathbf{2}}^{SU(2)} = t + t^{-1}$. $b_i \equiv e^{-v_i}$, and $(1 - t^2 b_i^{\pm} b_j^{\pm})$ means that all 4 factors with different signs are multiplied. The convention on representations (e.g. primes) all follows [36]. The numerators are invariant under $SO(7) \times Sp(n_{\mathbf{8}})$ Weyl symmetry, being character sums. Since the denominators are products with all possible $\pm$ signs, they are also invariant under $SO(7)$ Weyl group which flips $b_i \to b_i^{-1}$. So $\hat{Z}_1^{n_{\mathbf{8}}}$ is invariant under the Weyl group of $SO(7) \times Sp(n_{\mathbf{8}})$.

We expect our quantum mechanics to work also at $n_{\mathbf{8}} = 4$. Here, the 1d Coulomb branch with nonzero $\phi_I$ has a continuum. There may appear extra contribution from this continuum to the index [22], apart from (4.1.29). (For conceptual simplicity, we consider the problem at zero FI parameter $\xi = 0$.) The extra contribution from the 1d Coulomb continuum is neutral in $SO(7)\times Sp(4)$. This is because the extra states in the 1d Coulomb branch come from the region with large $\phi_I$, where all $U(k)$ charged fields acquire large masses. The charged fields are those which see $SO(7) \times Sp(4)$. So the extra continuum does not see these charges. Here we shall only test the $SO(7) \times Sp(4)$ symmetry enhancements at $n_{\mathbf{8}} = 4$. So we simply ignore the extra contribution, and show that (4.1.29) exhibits $SO(7) \times Sp(4)$ Weyl symmetry. The result at $k = 1$, showing $SO(7) \times Sp(4)$ Weyl symmetry, is given by

$$\hat{Z}_1^{n_{\mathbf{8}}=4} = \prod_{i<j} \frac{t^2}{(1-t^2 b_i^{\pm} b_j^{\pm})} \left[\chi_{\mathbf{42}}^{Sp(4)} f_0(v) + \chi_{\mathbf{48}}^{Sp(4)} f_1(v) + \chi_{\mathbf{27}}^{Sp(4)} f_2(v) + \chi_{\mathbf{8}}^{Sp(4)} f_3(v) + f_4(v)\right] \tag{4.1.34}$$

with $f_{0,1,2,3}(v)$ given by (4.1.33), and

$$\begin{aligned} f_4(v) \ = \ & -\chi_{\mathbf{13}}^{SU(2)} + \chi_{\mathbf{11}}^{SU(2)}(\chi_{\mathbf{21}}^{SO(7)} - \chi_{\mathbf{7}}^{SO(7)}) + \chi_{\mathbf{9}}^{SU(2)}(\chi_{\mathbf{7}}^{SO(7)} - \chi_{\mathbf{21}}^{SO(7)} - \chi_{\mathbf{27}}^{SO(7)} + \chi_{\mathbf{35}}^{SO(7)} \\ & + \chi_{\mathbf{105}}^{SO(7)} - \chi_{\mathbf{189}}^{SO(7)} - 1) - \chi_{\mathbf{7}}^{SU(2)}\big(2\chi_{\mathbf{7}}^{SO(7)} - 2\chi_{\mathbf{21}}^{SO(7)} - 2\chi_{\mathbf{27}}^{SO(7)} + \chi_{\mathbf{35}}^{SO(7)} + \chi_{\mathbf{77}}^{SO(7)} \\ & + 2\chi_{\mathbf{105}}^{SO(7)} - \chi_{\mathbf{168'}}^{SO(7)} - \chi_{\mathbf{189}}^{SO(7)} - \chi_{\mathbf{294}}^{SO(7)} - \chi_{\mathbf{330}}^{SO(7)} + \chi_{\mathbf{378}}^{SO(7)} - 1\big) \\ & - \chi_{\mathbf{5}}^{SU(2)}\big( - 3\chi_{\mathbf{7}}^{SO(7)} + 3\chi_{\mathbf{21}}^{SO(7)} + 3\chi_{\mathbf{27}}^{SO(7)} - \chi_{\mathbf{35}}^{SO(7)} - 2\chi_{\mathbf{77}}^{SO(7)} - 4\chi_{\mathbf{105}}^{SO(7)} + 2\chi_{\mathbf{168'}}^{SO(7)} + \chi_{\mathbf{182}}^{SO(7)} \\ & - \chi_{\mathbf{3}}^{SU(2)}\big(3\chi_{\mathbf{7}}^{SO(7)} - 4\chi_{\mathbf{21}}^{SO(7)} - 4\chi_{\mathbf{27}}^{SO(7)} + 2\chi_{\mathbf{35}}^{SO(7)} + 3\chi_{\mathbf{77}}^{SO(7)} + 6\chi_{\mathbf{105}}^{SO(7)} - 3\chi_{\mathbf{168'}}^{SO(7)} - \chi_{\mathbf{182}}^{SO(7)} \\ & - 2\chi_{\mathbf{189}}^{SO(7)} - \chi_{\mathbf{294}}^{SO(7)} - 4\chi_{\mathbf{330}}^{SO(7)} + \chi_{\mathbf{378}}^{SO(7)} + 2\chi_{\mathbf{616}}^{SO(7)} + 2\chi_{\mathbf{693}}^{SO(7)} + \chi_{\mathbf{819}}^{SO(7)} - \chi_{\mathbf{825}}^{SO(7)} - \chi_{\mathbf{1560}}^{SO(7)} - 2 \\ & + \big(4\chi_{\mathbf{7}}^{SO(7)} - 4\chi_{\mathbf{21}}^{SO(7)} - 5\chi_{\mathbf{27}}^{SO(7)} + 2\chi_{\mathbf{35}}^{SO(7)} + 4\chi_{\mathbf{77}}^{SO(7)} + 7\chi_{\mathbf{105}}^{SO(7)} - 3\chi_{\mathbf{168'}}^{SO(7)} - \chi_{\mathbf{182}}^{SO(7)} - 3\chi_{\mathbf{189}}^{SO(7)} - 5\chi \\ & + 2\chi_{\mathbf{378}}^{SO(7)} + 2\chi_{\mathbf{616}}^{SO(7)} + 3\chi_{\mathbf{693}}^{SO(7)} + 2\chi_{\mathbf{819}}^{SO(7)} - \chi_{\mathbf{1617}}^{SO(7)} - \chi_{\mathbf{1911}}^{SO(7)} - 1\big)\ . \end{aligned}$$

Now we consider the instanton strings of 6d super-Yang-Mills theories with $SO(7)$ gauge group and matters in $\mathbf{8}$. The number $n_{\mathbf{8}}$ of hypermultiplets cannot be arbitrary, due to gauge anomalies [7,37]. Without matters in other representations, one should have $n_{\mathbf{8}} = 2$ [7]. Incidently, the 6d consistency requirement

$n_{\mathbf{8}} = 2$ is also reflected in our ADHM-like construction, uplifted to 2d for instanton strings. This comes from 2d $U(k)$ gauge anomaly cancelation. First consider the $SU(k)$ anomaly, proportional to $D_{\mathbf{R}} = \pm 2T(\mathbf{R})$ for right/left moving fermions. From $D_{\mathbf{k}} = 1$, $D_{\mathbf{adj}} = 2k$, $D_{\mathbf{sym}_2} = k + 2$, $D_{\mathbf{anti}_2} = k - 2$, one obtains

$$-2\cdot 2k+2\cdot 4\cdot 1+2\cdot 2k+4\cdot 1+2\cdot(k-2)-2\cdot(k+2)-n_{\mathbf{8}}\cdot 2\cdot 1 = 2(2-n_{\mathbf{8}})\ . \quad (4.1.36)$$

These terms come from fermions in the multiplets $(\lambda_0, \lambda)$, $(q, \tilde{q})$, $(a, \tilde{a})$, $\phi$, $(b, \tilde{b})$, $(\hat{\lambda}, \check{\lambda})$, $(\Psi_a, \tilde{\Psi}_a)$, respectively. The $SU(k)$ anomaly cancels only at $n_{\mathbf{8}} = 2$. The overall $U(1)$ anomaly is proportional to the square of charges. The net anomaly is given by

$$2\cdot 4k\cdot 1^2+4k\cdot 1^2+2\cdot\frac{k^2-k}{2}\cdot 2^2-2\cdot\frac{k^2+k}{2}\cdot 2^2-n_{\mathbf{8}}\cdot 2k\cdot 1^2 = 2k(2-n_{\mathbf{8}})\ . \quad (4.1.37)$$

This again cancels at $n_{\mathbf{8}} = 2$. So our ADHM-like quiver consistently uplifts to 2d at $n_{\mathbf{8}} = 2$.

As a basic test of our 2d gauge theories, we study the 't Hooft anomalies of global symmetries. The full 2d symmetry is expected to be $SO(7) \times Sp(2) \times SU(2)_l \times SU(2)_r \times SU(2)_R$. From our UV description, we can only study $SU(4) \times U(2) \times SU(2)_l \times U(1)_J$. There is an alternative way of computing the anomalies on the strings, using anomaly inflow [5, 38]. By comparing two calculations, we shall provide a test of our gauge theories.

Using the inflow method, the 2d anomaly can be computed as follows. We first compute the anomaly polynomial 8-form of the 6d SCFT with a tensor multiplet, $SO(7)$ vector multiplet, and half-hypermultiplets in $\frac{1}{2}(\mathbf{8}, \mathbf{4})$ of $SO(7) \times Sp(2)$. The anomaly polynomial in the tensor branch consists of 1-loop contribution $I_{\text{1-loop}}$, coming from massless tensor/vector/hyper-multiplets, and the classical Green-Schwarz contribution $I_{GS}$ [39, 40]. The two contributions should partly cancel for the terms containing $SO(7)$ gauge fields [41,42]. $I_{\text{1-loop}}$ is given by

$$\begin{aligned} I_{\text{1-loop}} &= -\frac{3}{32}\left[\text{Tr}(F_{SO(7)}^2)\right]^2 + \frac{1}{16}\text{Tr}(F_{SO(7)}^2)\left[2\text{tr}_{\mathbf{4}}(F_{Sp(2)}^2) - 20c_2(R) - p_1(T)\right] + \cdots \\ &= -\frac{3}{2}\left[\frac{1}{4}\text{Tr}(F_{SO(7)}^2) + \frac{1}{12}\left(20c_2(R) + p_1(T) - 2\text{tr}_{\mathbf{4}}(F_{Sp(2)}^2)\right)\right]^2 + \cdots \ , \end{aligned} \tag{4.1.38}$$

where $\cdots$ denote terms independent of the $SO(7)$ field strength $F_{SO(7)}$. Following [42], we use the notation $\text{Tr} \equiv \frac{1}{h^\vee}\text{tr}_{\mathbf{adj}}$, and $\text{tr}_{\mathbf{adj}}(F^4) = -\text{tr}_{\text{fund}}(F^4) + 3(\text{Tr}(F^2))^2$, $\text{tr}_{\mathbf{8}}(F^4) = -\frac{1}{2}\text{tr}_{\text{fund}}(F^4) + \frac{3}{8}(\text{tr}(F^2))^2$, $\text{tr}_{\mathbf{8}}(F^2) = \text{Tr}(F^2)$, $\text{tr}_{\mathbf{adj}}(F^2) = 5\text{Tr}(F^2)$ for $SO(7)$. To cancel the 1-loop $SO(7)$ anomaly, one should have the following Green-Schwarz 8-form [42]:

$$I_{GS} = \frac{3}{2}I^2 \ , \quad I \equiv \frac{1}{4}\text{Tr}(F_{SO(7)}^2) + \frac{1}{12}\left(20c_2(R) + p_1(T) - 2\text{tr}_{\mathbf{4}}(F_{Sp(2)}^2)\right) \ . \tag{4.1.39}$$

This takes the form of $I_{GS} = \frac{1}{2}\Omega^{ij}I_iI_j$ with $i,j$ running over just 1, so that $I_1 = I$ and $\Omega^{11} = 3$. $\Omega^{11}$ may be fixed from the fact that it comes from $O(-3) \to \mathbb{P}^1$ geometry in F-theory, with self intersection number of $\mathbb{P}^1$ being 3. Knowing $I_i$ appearing in $I_{GS} = \frac{1}{2}\Omega^{ij}I_iI_j$, one can determine the 2d anomaly 4-form on the strings, from inflow. The formula is [5,38]

$$I_4 = -\Omega^{ij}k_i\left[I_j + \frac{1}{2}k_j\chi(T_4)\right] \ , \tag{4.1.40}$$

where $k_i$ is the string number in the $i$'th gauge group (or $i$'th tensor multiplet). We decomposed the 6d tangent bundle $T$ to $T_2 \times T_4$, along/normal to the strings. From this formula, one finds

$$I_4 = -\frac{3}{2}k^2\chi(T_4) - 3k\left[\frac{1}{4}\text{Tr}(F_{SO(7)}^2) + \frac{5}{3}c_2(R) + \frac{p_1(T)}{12} - \frac{1}{6}\text{tr}_{\mathbf{4}}(F_{Sp(2)}^2)\right] \tag{4.1.41}$$

for the $SO(7)$ instanton strings at $n_{\mathbf{8}} = 2$, with topological number $k$.

Now we compute $I_4$ from our gauge theory. A chiral fermion's anomaly 4-form is given by

$$I_4 = \pm \left[ \frac{1}{2} \mathrm{tr}(F^2) + \frac{p_1(T_2)}{24} \right] , \tag{4.1.42}$$

where $\pm$ signs are for left/right-moving fermions, respectively, in our convention. $F$ collectively denotes all background gauge fields for the global symmetries acting on the fermion. Here it is for $SU(4) \times U(1)_J \times SU(2)_l \times U(2)_F$. We can only study the anomalies of the symmetries surviving in UV, and check the consistency with (4.1.41). Fermi and vector multiplets have left-moving fermions, while chiral multiplets have right-moving fermions. Each multiplet contributes to terms of the form (4.1.42) with a suitable sign. Firstly, contributions from fields neutral in $SU(4) \times U(2)$ are already computed in [5]:

$$\begin{aligned}
(\lambda_0, \lambda) + (a, \tilde{a}) \ &: \ k^2(c_2(r) - c_2(l)) \\
\hat{\lambda}, \check{\lambda} \ &: \ (k^2 + k) \left[ \frac{F_R^2}{8} + \frac{c_2(r)}{2} + \frac{p_1(T_2)}{24} \right] \\
b, \tilde{b} \ &: \ -(k^2 - k) \left[ \frac{F_R^2}{8} + \frac{c_2(l)}{2} + \frac{p_1(T_2)}{24} \right] .
\end{aligned} \tag{4.1.43}$$

$R$ is the $U(1)$ Cartan of $SU(2)_R$. Here and later, we shall often use expressions like $c_2(r)$, $c_2(R) = \frac{F_R^2}{4}$ assuming symmetry enhancement, but only the $U(1)_J$ part is to be kept in UV. Namely, one first keeps the Cartan parts of the field strengths, for $J_r$, $J_R$. Then they are all replaced by $J$ and its field strength $F_J$. We present the results using $c_2(R)$ and $c_2(r)$ since this may suggest possible patterns of IR symmetry enhancement. (See also [5].) The fields charged under $SU(4) \times U(2)$ contribute to $I_4$ as follows:

$$\begin{aligned}
q, \tilde{q}, \phi \ &: \ -3k \left[ \frac{1}{4} \mathrm{Tr}(F_{SU(4)}^2) + 4 \cdot \frac{F_R^2}{8} + 4 \cdot \frac{p_1(T_2)}{24} \right] \\
\Psi_a, \tilde{\Psi}_a \ &: \ 2k \left[ \frac{1}{2} \mathrm{tr}_{\mathbf{2}}(F_{U(2)}^2) + 2 \cdot \frac{p_1(T_2)}{24} \right] .
\end{aligned} \tag{4.1.44}$$

Adding all, and using $p_1(T) = p_1(T_2) - 2c_2(l) - 2c_2(r)$, one obtains

$$I_4 = \frac{3}{2}k^2(c_2(r) - c_2(l)) - 3k\left[\frac{1}{4}\mathrm{Tr}(F^2_{SU(4)}) + \frac{5}{3}\cdot\frac{F_R^2}{4} + \frac{p_1(T)}{12} - \frac{1}{3}\mathrm{tr}_{\mathbf{2}}(F^2_{U(2)})\right]. \tag{4.1.45}$$

Here and below, we shall frequently use the fact that $\mathrm{Tr}(F^2)$ remains the same after restricting $F$ to a subalgebra if a long root of the original algebra is kept, so that unit instanton charge $\frac{1}{4}\int \mathrm{Tr}(F^2)$ remains the same [42]. Here it applies to $SU(4) \subset SO(7)$. As for $U(2) \subset Sp(2)$, or more generally $U(n) \subset Sp(n)$, the embedding is such that $\mathrm{tr}_{\mathbf{2n}} \to 2\mathrm{tr}_{\mathbf{n}}$. Taking these into account, (4.1.45) agrees with (4.1.41) upon restricting (4.1.41) to $SU(4)$, $U(2)$, $SU(2)_r \times SU(2)_R \to J$, and using $\chi(T_4) = c_2(l) - c_2(r)$. Their mixed anomalies with $U(k)$ also vanish.

One can study the elliptic genera $Z_k$ of $k$ instanton strings, whose spatial direction wraps $S^1$. The definition is almost identical to (4.1.15), except that there is another factor $e^{2\pi i\tau P}$ inside the trace, where $P$ is the left-moving momentum on $S^1$. The basic formula is given in [28, 29]. The result is obtained by simply replacing all $2\sinh$ functions in (4.1.16), (4.1.28), (4.1.29) by $2\sinh\frac{z}{2} \to \frac{i\theta_1(\tau|\frac{z}{2\pi i})}{\eta(\tau)} \equiv \theta(z)$. For instance, at $k = 1$, one obtains

$$Z_1(\tau, \epsilon_{1,2}, v_i, m_a) = \frac{1}{\theta(\epsilon_{1,2})}\sum_{i=1}^{4}\frac{\theta(4\epsilon_+ - 2v_i)\prod_{a=1}^{2}\theta(m_a \pm (v_i - \epsilon_+))}{\prod_{j(\neq i)}\theta(v_{ij})\theta(2\epsilon_+ - v_{ij})\theta(2\epsilon_+ - v_i - v_j)} \tag{4.1.46}$$

where $v_{ij} \equiv v_i - v_j$. Some tests of these formulae will be given in section 4.2.

### 4.1.2 $G_2$ instantons and matters in 7

With a hypermultiplet in **8**, one can Higgs $SO(7)$ to $G_2$. Decomposing the scalar to $\mathbf{8} \to \mathbf{7}\oplus\mathbf{1}$ in $G_2$, **1** is given VEV and decouples in IR. **7** is eaten up by the broken part of the $SO(7)$ gauge fields, since $\mathbf{21} \to \mathbf{14}\oplus\mathbf{7}$. The matter consists of two half hypermultiplets, forming a doublet of flavor symmetry $Sp(1)_F$. The scalar can be written as $[\Phi_{Aa}]_{(s_1,s_2,s_3)}$, where $A = 1, 2$ is the doublet index of $SU(2)_R$

R-symmetry, $a=1,2$ is that of $Sp(1)_F$, and $s_{1,2,3}=\pm\frac{1}{2}$ label the components of $\mathbf{8}$. It satisfies the reality condition $(\Phi^*)^{Aa}_{(s_1,s_2,s_3)}=\epsilon^{AB}\epsilon^{ab}(\Phi_{Bb})_{(s_1,s_2,s_3)}$. Let us take $\Phi_{Aa}\equiv[\Phi_{Aa}]_{(+,+,+)}+[\Phi_{Aa}]_{(-,-,-)}$, satisfying $(\Phi^*)^{Aa}=\epsilon^{AB}\epsilon^{ab}\Phi_{Bb}$. One takes $\Phi_{Aa}=\epsilon_{Aa}\Phi$, with a pure imaginary VEV $\Phi$. This preserves a diagonal subgroup of $SU(2)_R\times Sp(1)_F$, which is the $SU(2)_R$ symmetry after Higgsing. At general $n_{\mathbf{8}}$, we give VEV to the last hypermultiplet scalar, $a=n_{\mathbf{8}}$. One should lock the chemical potentials as

$$m_{\text{last}}-\epsilon_+\pm v_4=0 \tag{4.1.47}$$

with both signs, not to rotate the scalar VEV. So we should take $m_{\text{last}}-\epsilon_+=0$, $v_4=0$. The former condition turns off the $Sp(1)\subset Sp(n_{\mathbf{8}})$ chemical potential $m_{\text{last}}$, and the latter reduces the rank of gauge group by 1. As the index is invariant under the RG flow triggered by the scalar VEV, one can get the IR $G_2$ index by constraining the $SO(7)$ index.

In our $SU(4)$ formalism, the bulk scalars are written as $Q_i,\tilde{Q}^i$, where $i=1,2,3,4$. Giving VEV to $\mathbf{1}$ amounts to turning on $Q_4=\tilde{Q}^4=M\neq 0$ (real), where we take the unbroken $SU(3)\subset G_2$ to be labeled by $i=1,2,3$. In 1d, the background fields couple to the 1d fields as

$$J_{\Psi_{\text{last}}}\sim Q_i\tilde{q}^i\ ,\quad J_{\tilde{\Psi}_{\text{last}}}\sim\tilde{Q}^iq_i\ . \tag{4.1.48}$$

The second potential $|J_{\tilde{\Psi}_{\text{last}}}|^2\sim M^2|q_4|^2$ gives mass to $q_4$, while the first one gives mass to $\tilde{q}^4$.[4] The $SU(4)$ ADHM fields reduce to the $SU(3)$ ADHM fields at low energy. Among the extra fields, $\phi_i$ with $i=1,2,3,4$ decomposes into $\phi_i$ with $i=1,2,3$ in $(\bar{\mathbf{k}},\bar{\mathbf{3}})$, and $\phi_4$ in $(\bar{\mathbf{k}},\mathbf{1})$. If $n_{\mathbf{8}}\geq 2$, one still has $n_{\mathbf{7}}=n_{\mathbf{8}}-1$ pairs of Fermi multiplets $\Psi_a,\tilde{\Psi}_a$ left in $(\mathbf{k},\mathbf{1})+(\bar{\mathbf{k}},\mathbf{1})$, $a=1,\cdots,n_{\mathbf{7}}$. To summarize,

[4]One may more generally take $J_\Psi\sim\alpha Q_i\tilde{q}^i+\beta\tilde{Q}^i\phi_i$, compatible with $U(k)\times SU(4)$. However, with $SU(4)$ broken to $SU(3)$, $\tilde{q}^4$ and $\phi_4$ have same charges in unbroken symmetries, and $\alpha,\beta$ does not affect the IR physics.

one first has the $SU(3)$ ADHM fields,

$$\begin{aligned}
A_\mu, \lambda_0, \lambda &: \ \mathcal{N}=(0,4)\ U(k) \text{ vector multiplet} \\
q_i, \tilde{q}^i &: \ (\mathbf{k}, \bar{\mathbf{3}}) + (\bar{\mathbf{k}}, \mathbf{3}) \quad (i=1,2,3) \\
a, \tilde{a} &: \ (\mathbf{adj}, \mathbf{1})\ .
\end{aligned} \tag{4.1.49}$$

In addition, one has

$$\begin{aligned}
\phi_i, \phi_4 &: \ \text{chiral in } (\bar{\mathbf{k}}, \bar{\mathbf{3}})_{J=\frac{1}{2}} + (\bar{\mathbf{k}}, \mathbf{1})_{J=\frac{1}{2}} \\
b, \tilde{b} &: \ \text{chiral in } (\overline{\mathbf{anti}}_2, \mathbf{1})_{J=\frac{1}{2}} \\
\hat{\lambda} &: \ \text{Fermi in } (\mathbf{sym}_2, \mathbf{1})_{J=0} \\
\check{\lambda} &: \ \text{Fermi in } (\mathbf{sym}_2, \mathbf{1})_{J=-1}\ .
\end{aligned} \tag{4.1.50}$$

For $n_{\mathbf{7}} \leq 3$ hypermultiplet matters in representation $\mathbf{7}$, there are extra Fermi multiplets:

$$\Psi_a\ ,\ \tilde{\Psi}_a\ :\ (\mathbf{k}, \mathbf{1}) + (\bar{\mathbf{k}}, \mathbf{1})\ \ , \qquad a = 1, \cdots, n_{\mathbf{7}}\ . \tag{4.1.51}$$

The $\mathcal{N}=(0,1)$ action follows from a construction similar to $SO(7)$ in section 2.1.

The index $Z_k$ for $k$ $G_2$ instantons can either be obtained from the Witten index of the above gauge theory, or by taking the Higgsing condition of the $SO(7)$ index, $m_{n_{\mathbf{8}}} = \epsilon_+$, $v_4 = 0$. It may be more illustrative to write both the contour integral expression and the residue sum. The contour integral expression for the index is given by

$$Z_k = \frac{1}{k!}\oint \prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}\cdot\frac{\prod_{I\neq J}2\sinh\frac{\phi_{IJ}}{2}\cdot\prod_{I,J}2\sinh\frac{2\epsilon_+-\phi_{IJ}}{2}}{\prod_{I=1}^{k}\prod_{i=1}^{3}2\sinh\frac{\epsilon_+\pm(\phi_I-v_i)}{2}\cdot\prod_{I,J}2\sinh\frac{\epsilon_{1,2}+\phi_{IJ}}{2}}$$
$$\times\frac{\prod_{I\leq J}\left(2\sinh\frac{\phi_I+\phi_J}{2}\cdot 2\sinh\frac{\phi_I+\phi_J-2\epsilon_+}{2}\right)}{\prod_I\left(\prod_{i=1}^{3}2\sinh\frac{\epsilon_+-\phi_I-v_i}{2}\cdot 2\sinh\frac{\epsilon_+-\phi_I}{2}\right)\cdot\prod_{I<J}2\sinh\frac{\epsilon_{1,2}-\phi_I-\phi_J}{2}}$$
$$\cdot\prod_{I=1}^{k}\prod_{a=1}^{n_{\mathbf{7}}}2\sinh\frac{m_a\pm\phi_I}{2} \tag{4.1.52}$$

The residue sum, labeled by $SU(3)$ colored Young diagrams, is given by[5]

$$Z_k = (-1)^k\sum_{\vec{Y};|\vec{Y}|=k}\prod_{i=1}^{3}\prod_{s\in Y_i}\frac{2\sinh(\phi(s))\cdot 2\sinh(\epsilon_+-\phi(s))}{\prod_{j=1}^{3}\left(2\sinh\frac{E_{ij}(s)-\epsilon_+\pm\epsilon_+}{2}\cdot 2\sinh\frac{\epsilon_+-\phi(s)-v_j}{2}\right)\cdot 2\sinh\frac{\epsilon_+-\phi(s)}{2}}$$
$$\cdot\prod_{i\leq j}^{3}\prod_{s_{i,j}\in Y_{i,j};s_i<s_j}\frac{2\sinh\frac{\phi(s_i)+\phi(s_j)-\epsilon_+\pm\epsilon_+}{2}}{2\sinh\frac{\epsilon_{1,2}-\phi(s_i)-\phi(s_j)}{2}}\cdot\prod_{i=1}^{3}\prod_{s\in Y_i}\prod_{a=1}^{n_{\mathbf{7}}}2\sinh\frac{m_a\pm\phi(s)}{2} \tag{4.1.53}$$

where $\phi(s)$ and $E_{ij}(s)$ are defined in (4.1.17), (4.1.19).

We first study the case with $n_{\mathbf{7}}=0$. We can test the results against known $G_2$ instanton partition functions of [43]. We tested (4.1.53) till $k\leq 3$. Firstly, at $k=1$, it will be illustrative to make a basic presentation, directly from the contour integral. (4.1.52) at $k=1$ is given by

$$\left(2\sinh\frac{\epsilon_{1,2}}{2}\right)Z_1 = \oint d\phi\frac{2\sinh\epsilon_+\cdot 2\sinh\phi\cdot 2\sinh(\phi-\epsilon_+)}{\prod_{i=1}^{3}\left(2\sinh\frac{\epsilon_+\pm(\phi-v_i)}{2}\cdot 2\sinh\frac{\epsilon_+-\phi-v_i}{2}\right)\cdot 2\sinh\frac{\epsilon_+-\phi}{2}}\,. \tag{4.1.54}$$

At $\eta>0$, the poles are chosen at $\phi=v_i-\epsilon_+$, $i=1,2,3$. So one obtains

$$\left(2\sinh\frac{\epsilon_{1,2}}{2}\right)Z_1 = \sum_{i=1}^{3}\frac{2\sinh(v_i-2\epsilon_+)}{\prod_{j(\neq i)}\left(2\sinh\frac{v_{ij}}{2}\cdot 2\sinh\frac{2\epsilon_+-v_{ij}}{2}\cdot 2\sinh\frac{2\epsilon_++v_j}{2}\right)\cdot 2\sinh\frac{v_i-2\epsilon_+}{2}} \tag{4.1.55}$$

[5]The factors on the first line of (4.1.53) containing $E_{ij}$ are the residues for pure $SU(3)$ theory. In this type of expression, one finds an overall $(-1)^{Nk}$ factor for pure $SU(N)$ instantons. This is why we have $(-1)^k$ in (4.1.53).

where we used $v_1 + v_2 + v_3 = 0$. Each residue only exhibits Weyl symmetry of $SU(3)$, given by 3! permutations of $v_1, v_2, v_3$. However, the sum of three residues exhibits enhanced Weyl symmetry of $G_2$, the dihedral group $D_6$ of order 12. The extra transformation generating full $D_6$ is $v_i \to -v_i$ for all $i = 1, 2, 3$, $SU(3)$ charge conjugation. One can show that $Z_1$ is given by

$$\left(2\sinh\frac{\epsilon_{1,2}}{2}\right) Z_1 = \frac{t^3(1+t^2)(1+t^2\chi_{\mathbf{7}}^{G_2}(v)+t^4)}{\prod_{i<j}(1-t^2e^{v_{ij}})(1-t^2e^{-v_{ij}})} = t^3\sum_{n=0}^{\infty}\chi_{(0,n)}^{G_2}(v)t^{2n} \ , \tag{4.1.56}$$

where $t \equiv e^{-\epsilon_+}$. $\chi_{\mathbf{7}}^{G_2} = 1+\chi_{\mathbf{3}}^{SU(3)}+\chi_{\bar{\mathbf{3}}}^{SU(3)} = 1+\sum_{i=1}^{3}(e^{v_i}+e^{-v_i})$ is the character of $\mathbf{7}$. $\chi_{(0,n)}^{G_2}$ is the character of the irrep $(0, n)$ of $G_2$, which is the $n$'th symmetric product of the adjoint representation $\mathbf{14}$. (4.1.56) is known as the correct $G_2$ instanton partition function at $k = 1$ [43].

At $k = 2$, $Z_2$ can be rearranged into (where $t = e^{-\epsilon_+}$, $u = e^{-\epsilon_-}$)

$$\left(\prod_{n=1}^{2}2\sinh\frac{n\epsilon_{1,2}}{2}\right)Z_2 = \frac{t^{24}}{\prod_{i<j}(1-t^2e^{\pm v_{ij}})(1-t^3u^{\pm\frac{1}{2}}e^{\pm v_{ij}})}\left[\chi_{\mathbf{20}}^{SU(2)}+\sum_{n=1}^{18}\chi_{\mathbf{n}}^{SU(2)}g_n(v_i,u)\right] , \tag{4.1.57}$$

where $SU(2)$ is still diag$[SU(2)_r \times SU(2)_R]$, and $g_n(v_i, u)$'s are given by

$$\begin{aligned}
g_{18} &= \chi_{\mathbf{7}}^{G_2}+1 \ , \quad g_{17} = \chi_{\mathbf{2}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}+1) \ , \\
g_{16} &= \chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{27}}^{G_2}+1 \ , \quad g_{15} = \chi_{\mathbf{2}}^{SU(2)_l}(3\chi_{\mathbf{7}}^{G_2}+1) \ , \\
g_{14} &= \chi_{\mathbf{3}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}+1)+2\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{27}}^{G_2}+1 \ , \quad g_{13} = \chi_{\mathbf{2}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{14}}^{G_2}+\chi_{\mathbf{27}}^{G_2}-\chi_{\mathbf{64}}^{G_2}+2) \ , \\
g_{12} &= \chi_{\mathbf{3}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{14}}^{G_2}-\chi_{\mathbf{64}}^{G_2}+2)+2\chi_{\mathbf{7}}^{G_2}+2\chi_{\mathbf{14}}^{G_2}-2\chi_{\mathbf{64}}^{G_2}+2 \\
g_{11} &= \chi_{\mathbf{4}}^{SU(2)_l}+\chi_{\mathbf{2}}^{SU(2)_l}(2\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{64}}^{G_2}-\chi_{\mathbf{189}}^{G_2}+1) \\
g_{10} &= \chi_{\mathbf{3}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{14}}^{G_2}-\chi_{\mathbf{77}}^{G_2})+3\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{14}}^{G_2}-\chi_{\mathbf{64}}^{G_2}-2\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{189}}^{G_2}+1 \\
g_9 &= \chi_{\mathbf{4}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{14}}^{G_2})+\chi_{\mathbf{2}}^{SU(2)_l}(3\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{14}}^{G_2}-\chi_{\mathbf{64}}^{G_2}-3\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}+1) \\
g_8 &= \chi_{\mathbf{3}}^{SU(2)_l}(\chi_{\mathbf{77}}^{G_2}-\chi_{\mathbf{14}}^{G_2})+2\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{64}}^{G_2}-\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{189}}^{G_2}+\chi_{\mathbf{378}}^{G_2}+1
\end{aligned} \tag{4.1.58}$$

$$
\begin{aligned}
g_7 &= \chi_{\mathbf{4}}^{SU(2)_l}(\chi_{\mathbf{77}}^{G_2}-\chi_{\mathbf{7}}^{G_2})+\chi_{\mathbf{2}}^{SU(2)_l}(2\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{64}}^{G_2}+2\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{286}}^{G_2}+\chi_{\mathbf{448}}^{G_2}+1)\\
g_6 &= \chi_{\mathbf{3}}^{SU(2)_l}(\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{64}}^{G_2}+\chi_{\mathbf{189}}^{G_2}+1)+\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{14}}^{G_2}+\chi_{\mathbf{27}}^{G_2}-\chi_{\mathbf{64}}^{G_2}+\chi_{\mathbf{182}}^{G_2}\\
&\quad -\chi_{\mathbf{286}}^{G_2}+\chi_{\mathbf{378}}^{G_2}+\chi_{\mathbf{448}}^{G_2}+1\\
g_5 &= \chi_{\mathbf{4}}^{SU(2)_l}(\chi_{\mathbf{77}}^{G_2}-2\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{14}}^{G_2}-1)\\
&\quad +\chi_{\mathbf{2}}^{SU(2)_l}(-\chi_{\mathbf{7}}^{G_2}+2\chi_{\mathbf{14}}^{G_2}+\chi_{\mathbf{27}}^{G_2}-2\chi_{\mathbf{64}}^{G_2}+2\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}+\chi_{\mathbf{189}}^{G_2}-\chi_{\mathbf{286}}^{G_2}+\chi_{\mathbf{378}}^{G_2}+2)\\
g_4 &= -\chi_{\mathbf{5}}^{SU(2)_l}\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{3}}^{SU(2)_l}(2\chi_{\mathbf{77}}^{G_2}-\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{14}}^{G_2}-\chi_{\mathbf{27}}^{G_2}-2\chi_{\mathbf{64}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{286}}^{G_2}+1)\\
&\quad +2\chi_{\mathbf{14}}^{G_2}+\chi_{\mathbf{27}}^{G_2}-\chi_{\mathbf{64}}^{G_2}+\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{273}}^{G_2}-2\chi_{\mathbf{286}}^{G_2}+\chi_{\mathbf{448}}^{G_2}+1\\
g_3 &= -\chi_{\mathbf{4}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{27}}^{G_2}+\chi_{\mathbf{64}}^{G_2})+\chi_{\mathbf{2}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}-\chi_{\mathbf{14}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{189}}^{G_2}-\chi_{\mathbf{286}}^{G_2}-\chi_{\mathbf{729}}^{G_2})\\
g_2 &= -\chi_{\mathbf{5}}^{SU(2)_l}(\chi_{\mathbf{7}}^{G_2}+\chi_{\mathbf{14}}^{G_2}+1)+\chi_{\mathbf{3}}^{SU(2)_l}(\chi_{\mathbf{64}}^{G_2}-4\chi_{\mathbf{14}}^{G_2}-2\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{189}}^{G_2}-\chi_{\mathbf{448}}^{G_2}-2)\\
&\quad +2\chi_{\mathbf{7}}^{G_2}-2\chi_{\mathbf{14}}^{G_2}+2\chi_{\mathbf{64}}^{G_2}-3\chi_{\mathbf{77}}^{G_2}-\chi_{\mathbf{273}}^{G_2}-\chi_{\mathbf{729}}^{G_2}-1\\
g_1 &= \chi_{\mathbf{4}}^{SU(2)_l}(2\chi_{\mathbf{7}}^{G_2}-2\chi_{\mathbf{14}}^{G_2}-\chi_{\mathbf{27}}^{G_2}+\chi_{\mathbf{64}}^{G_2}-2\chi_{\mathbf{77}}^{G_2}-1)\\
&\quad +\chi_{\mathbf{2}}^{SU(2)_l}(2\chi_{\mathbf{7}}^{G_2}-3\chi_{\mathbf{14}}^{G_2}+2\chi_{\mathbf{64}}^{G_2}-4\chi_{\mathbf{77}}^{G_2}+\chi_{\mathbf{182}}^{G_2}-\chi_{\mathbf{273}}^{G_2}-\chi_{\mathbf{448}}^{G_2}-1)\ . \qquad (4.1.59)
\end{aligned}
$$

As the numerator is manifestly arranged into $G_2$ characters, it shows enhanced $G_2$ Weyl symmetry. The denominator is also invariant under the extra generator $v_i \to -v_i$ of $D_6$, being invariant under $G_2$ Weyl symmetry. One can also check the agreement with the known $G_2$ partition function at $k=2$. For the simplicity of comparison, let us turn off all $v_i = 0$ and $\epsilon_- = 0$. Then, (4.1.57) becomes

$$
\begin{aligned}
\left(2\sinh\frac{\epsilon_{1,2}}{2}\right) Z_2 &= \frac{t^7}{(1-t)^{14}(1+t)^8(1+t+t^2)^7}\big[1+t+10t^2+31t^3+75t^4\\
&\quad +180t^5+385t^6+637t^7+975t^8+1360t^9+1614t^{10}\\
&\quad +1666t^{11}+1614t^{12}+\cdots+t^{22}\big] \qquad (4.1.60)
\end{aligned}
$$

where the omitted terms $\cdots$ can be restored by the $t \to t^{-1}$ Weyl symmetry of $SU(2)$ (i.e. the coefficients of $t^p$ and $t^{22-p}$ are same on the numerator). The overall $t^7$ factor is like a zero point energy factor, and is needed to have this Weyl symmetry. Apart from this factor, (4.1.60) agrees with eqn.(9.5) of [44] after correcting a typo there, as noted in [43].

At $k=3$, we only show the simplified form of (4.1.53) at $v_i=0$, $\epsilon_-=0$, which is

$$\begin{aligned}\left(2\sinh\frac{\epsilon_{1,2}}{2}\right)Z_3 &= \frac{t^{11}}{(1-t)^{22}(1+t)^{14}(1+t^2)^7(1+t+t^2)^9}\big[1+t+11t^2+34t^3\\ &\quad +124t^4+352t^5+1055t^6+2657t^7+6584t^8+14635t^9\\ &\quad +31194t^{10}+61229t^{11}+114367t^{12}+198932t^{13}+329172t^{14}\\ &\quad +511194t^{15}+755093t^{16}+1051845t^17+1394817t^{18}\\ &\quad +1749632t^{19}+2091341t^{20}+2368619t^{21}+2557449t^{22}\\ &\quad +2619060t^{23}+2557449t^{24}+\cdots+t^{46}\big]\ ,\end{aligned}\tag{4.1.61}$$

where $\cdots$ can again be restored by noting that coefficients of $t^p$ and $t^{46-p}$ are same on the numerator. Apart from the overall $t^{11}$ factor which guarantees Weyl symmetry, this again agrees with eqn.(4.16) of [43]. Although we did comparisons till $k=3$, one can in principle continue to test for higher $k$'s whether our (4.1.53) agrees with the results of [43].

Now as for the indices at $n_{\mathbf{7}}\geq 1$, these observables have not been computed or studied in the literature, to the best of our knowledge. Here we simply note that, making expansions of the indices in $t=e^{-\epsilon_+}$, one observes that the coefficients are characters of $G_2\times Sp(n_{\mathbf{7}})$. At least at $k=1$, this does not need independent calculations, since we already illustrated the symmetry enhancement of $SO(7)\times Sp(n_{\mathbf{8}})$ in the previous subsection. Also, whenever we provide concrete tests of some $SO(7)$ results in section 3 and section 4, this implies similar tests of the $G_2$ results at $n_{\mathbf{7}}=n_{\mathbf{8}}-1$ by Higgsing.

At $n_{\mathbf{7}}=1$, 6d SCFT exists with $G_2$ gauge group. This can be obtained from 6d $SO(7)$ theory at $n_{\mathbf{8}}=2$ by Higgsing. Our 2d gauge theories on $G_2$ instantons can also be uplifted to 2d gauge theories. As in the previous subsection, this gauge theory is free of $U(k)$ gauge anomaly. The 2d anomaly of $G_2 \times Sp(n_{\mathbf{7}}) \times SU(2)_R \times SU(2)_r \times SU(2)_l$ global symmetries, computed from anomaly inflow, is also compatible with the $SU(3) \times U(n_{\mathbf{7}}) \times U(1)_J \times SU(2)_l$ anomalies of our 2d gauge theories. To see this, one first restricts $F_{SO(7)} \to F_{G_2}$ which leaves $\mathrm{Tr}(F^2)$ invariant, $\mathrm{tr}_{\mathbf{4}}(F^2_{Sp(2)}) \to \mathrm{tr}_{\mathbf{2}}(F^2_{Sp(1)}) + \mathrm{tr}_{\mathbf{2}}(F^2_{Sp(1)'})$, and note that $c_2(R) = \frac{1}{4}\mathrm{Tr}(F_R^2) = \frac{1}{2}\mathrm{tr}_{\mathrm{fund}}(F_R^2)$ [42]. Since we lock $Sp(1)'$ and $SU(2)_R$ during Higgsing, one identifies $F_{Sp(1)'} = F_R$. Then, both anomaly 4-forms (4.1.41), (4.1.45) reduce to

$$I_4 = -\frac{3}{2}k^2\chi(T_4) - 3k\left[\frac{1}{4}\mathrm{Tr}(F^2_{G_2}) + \frac{4}{3}c_2(R) + \frac{p_1(T)}{12} - \frac{1}{6}\mathrm{tr}_{\mathbf{2}}(F^2_{Sp(1)})\right] , \quad (4.1.62)$$

with restrictions to UV symmetry understood for gauge theory anomalies. So the inflow anomaly and 2d gauge theory anomaly continue to agree with each other.

The elliptic genera for the strings can be computed similarly. One takes the formulae (4.1.52) or (4.1.53), and replace $2\sinh\frac{z}{2} \to \frac{i\theta_1(\tau|\frac{z}{2\pi i})}{\eta(\tau)} \equiv \theta(z)$ for all $2\sinh$ functions. The $G_2$ symmetry of this elliptic genus at $k=1$ is systematically discussed in [20].

At $n_{\mathbf{7}} = 1$, one has a pair $\Psi, \tilde{\Psi}$ of Fermi multiplets. One can again investigate the effect of bulk Higgsing $G_2 \to SU(3)$. In the bulk, one decomposes $\mathbf{7} \to \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1}$, where scalar in $\mathbf{1}$ assumes VEV and breaks $G_2$ into $SU(3)$. The other hypermultiplet fields are eaten up by vector multiplets for the broken symmetry. The constant VEV of the bulk scalar $\equiv Q$ in $\mathbf{1}$ will behave as a background field in 1d/2d ADHM-like models. With foresight on the $SU(3)$ instantons studied in [5], we propose that the coupling of the background bulk field $Q$ to the $G_2$ ADHM-like gauge theory is given by

$$J_\Psi \sim Q\phi \ , \quad J_{\tilde{\Psi}} \sim Q\epsilon^{ijk} q_i \phi_j q_k^\dagger \ , \quad (4.1.63)$$

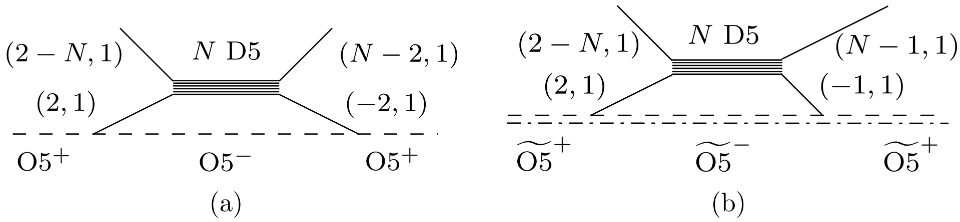



Figure 4.1: Brane realizations of (a) $SO(2N)$ and (b) $SO(2N+1)$ gauge theories

where $\phi \equiv \phi_4$. The $\mathcal{N} = (0,1)$ superpotential $J_{\tilde{\Psi}}$ is compatible with symmetries, but at this stage it may not be obvious why we should turn it on in this way. $\Psi$ and the chiral multiplet $\phi$ become massive due to $J_\Psi$, and decouple at low energy. However, $\tilde{\Psi}$ does not decouple at low energy, since it does not acquire mass. In fact, the remaining system (including $\tilde{\Psi}$, which was called $\zeta$ in [5]) with the above cubic superpotential was studied in [5], which showed various nontrivial physics of the $SU(3)$ instanton strings. In 1d, this provides a novel alternative ADHM-like description for $SU(3)$ instanton particles. In 2d, this is (by now) the uniquely known $SU(3)$ ADHM construction of instanton strings without matters. All models presented so far in this paper, for $SO(7)$ and $G_2$ instantons, were initially constructed by guessing the un-Higgsing procedures from $SU(3)$. See [5] for further discussions on the last $SU(3)$ model.

## 4.2 Exceptional instantons from D-branes

### 4.2.1 Brane setup and quantum mechanics

In this section, we test some indices of the previous section, using 5-brane webs for the 5d $\mathcal{N} = 1$ gauge theories with $SO(N)$ gauge groups and matters in spinor representations [45]. A type IIB 5-brane web on $x^5, x^6$ plane consists of $(p,q)$ 5-branes stretched along lines with slope $q/p$: e.g. D5-branes $(1,0)$ along $x^5$ and NS5-branes $(0,1)$ along $x^6$ directions. They occupy $x^0, \cdots, x^4$ direc-

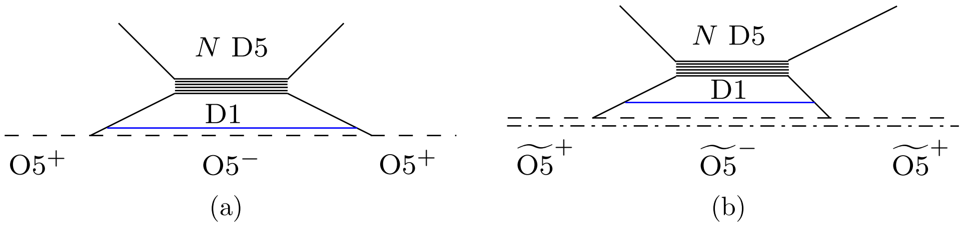



Figure 4.2: Instantons of (a) $SO(2N)$ and (b) $SO(2N+1)$ theories

tions for the 5d QFT. $SO(N)$ gauge theories are realized by 5-brane webs with orientifold 5-planes. An NS5-brane crossing the O5-plane bends to a suitable $(p,1)$-brane, and changes the types of O5 across NS5. An $SO(2N)$ theory is engineered by suspending $N$ D5-branes between two NS5-branes, also with an $O5^-$, as shown in Fig. 4.1(a). $SO(2N+1)$ theory is realized by $N$ D5-branes and an $\widetilde{O5}^-$ plane, which is an $O5^-$ with a half D5. See Fig. 4.1(b). Dashed-dotted line is a monodromy cut, to have $(p,q)$ 5-branes at right angles with properly quantized charges [45]. In these constructions, instanton particles are D1-branes stretched between two NS5-branes, as shown in Fig. 4.2. In this setting, a 5d hypermultiplet in the spinor representation is introduced as follows [45]. One introduces another NS5-brane as shown in Fig. 4.3. D1$'$-branes suspended between NS5$_1$ and NS5$_2$ are the particles obtained by quantizing the hypermultiplet in the $SO(N)$ spinor representation. (See [45] for the chirality of the $SO(2N)$ spinor.) The mass of this field is proportional to the distance between NS5$_1$ and NS5$_2$. To introduce two hypermultiplets in the spinor representation, one puts another NS5-brane on the right side, as shown in Fig. 4.4. Note that for $SO(N)$ gauge theory with $N \leq 6$, NS5$_1$ and NS5$_2$-branes do not intersect. For $N = 7, 8$, NS5$_1$ and NS5$_2$ are parallel to each other. In the last cases, there are extra continua of D1$'$ branes, orthogonally suspended between these parallel NS5-branes, which can escape to infinity and do not belong to 5d

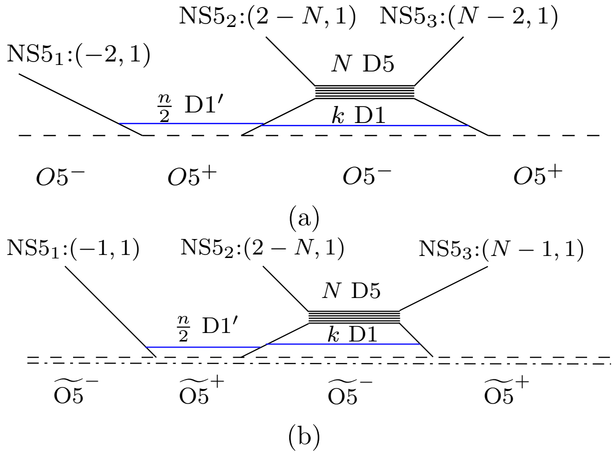



Figure 4.3: Hypermultiplet in the spinor representation of (a) $SO(2N)$ and (b) $SO(2N+1)$.

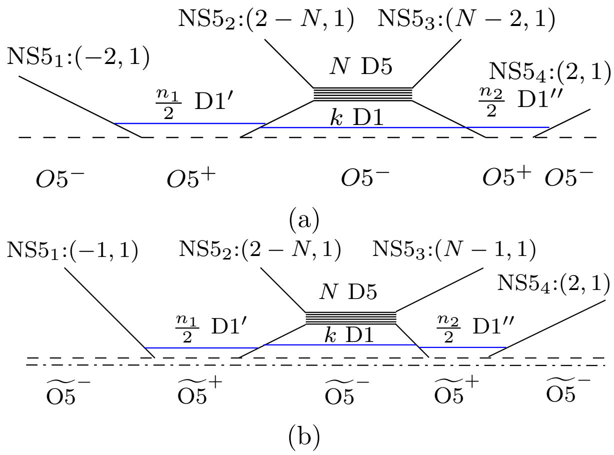



Figure 4.4: Two hypermultipets in the spinor representations of (a) $SO(2N)$ and (b) $SO(2N+1)$.

QFT. In section 4.2.2 we discuss this extra sector in more detail. When $N \geq 9$, NS5$_1$ and NS5$_2$ meet at a certain point. In this case, we do not know how to use this setting to study the 5d QFT. So in the rest of this paper, we focus on $SO(N)$ QFTs with $N \leq 8$.

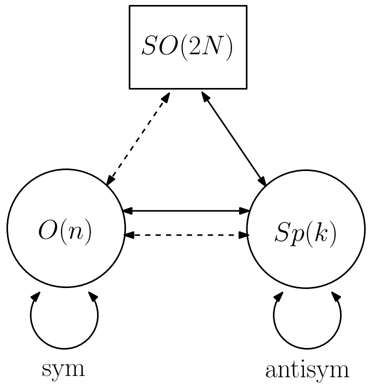
We discuss the quantum mechanical gauge theory, with given numbers of $SO(N)$ instantons $k$ and hypermultiplet particles $n_a$. Their Witten indices will be used to test some results of section 2. In section 2, we did not fix the numbers of hypermultiplet particles, but instead had chemical potentials $m_a$ for $Sp(n_{\mathbf{8}})$. Expanding the indices of section 2 in $e^{-m_a}$, the coefficients will be the indices with fixed $k, n_a$, studied in this section.

We start from the case with 1 hypermultiplet, and consider the quantum mechanics of the D1 and D1$'$ branes. We first explain the symmetries. There is $SO(4) \sim SU(2)_l \times SU(2)_r$ rotating $x^1, \cdots, x^4$, and $SO(3) \sim SU(2)_R$ rotating $x^7, x^8, x^9$. The quantum mechanics preserves 4 real SUSY $\bar{Q}^{\dot{\alpha}A}$, where $\dot{\alpha}$ and $A$ are doublet indices of $SU(2)_r$ and $SU(2)_R$. It can be regarded as the 1d reduction of 2d $\mathcal{N} = (0,4)$ SUSY. There are symmetries associated with D-branes and orientifolds. For $r$ D1's and $N$ D5's on various O5-planes, the symmetries are given as follows:

| branes | $O5^+$ | $O5^-$ | $\widetilde{O5}^+$ | $\widetilde{O5}^-$ |
|---|---|---|---|---|
| $N$ D5 | $Sp(N)$ | $SO(2N)$ | $Sp(N)$ | $SO(2N+1)$ |
| $r$ D1 | $O(2r)$ | $Sp(r)$ | $O(2r)$ | $Sp(r)$ |

Here $r$ is a half-integer $r = n/2$ for $O5^+$, $\widetilde{O5}^+$. So D1 and D1$'$ in Fig. 4.3 have $Sp(k) \times O(n)$ gauge symmetry, while D5's induce $SO(2N)$ or $SO(2N+1)$ global symmetry.

The quantum mechanical 'fields' are derived from open strings. They are shown in Fig. 4.5 and 4.6 for $SO(2N)$ and $SO(2N+1)$. The formal '$SO(1)$' in Fig. 4.6(a) comes from the half D5-brane on $\widetilde{\mathrm{O5}}^-$, on the left side of NS5$_1$ in Fig. 4.3. The Lagrangian of this system preserving $\mathcal{N} = (0,4)$ supersymmetry can be written down in a canonical manner. We focus on the bosonic part here. Along the strategy of [46], we first construct the Lagrangian in $\mathcal{N} = (0,2)$ formalism, specifying the two possible types of superpotentials $E$ and $J$ for



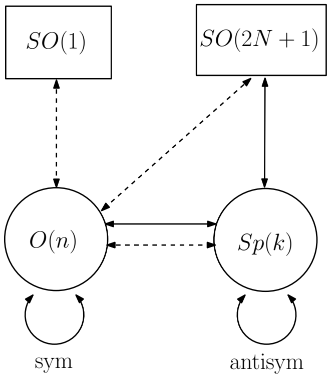

(a)

| Mode | Field | Type | $Sp(k)$ | $O(n)$ | $SO(2N)$ |
|---|---|---|---|---|---|
| D1–D1 | $(A_t, \varphi, \lambda^{\dot{\alpha}A})$ | vector | **sym** | – | – |
| | $(a_{\alpha\dot{\beta}}, \chi^A_\alpha)$ | hyper | **anti** | – | – |
| D1–D5 | $(q_{\dot{\alpha}}, \psi^A)$ | hyper | **2k** | – | **2N** |
| D1′–D1′ | $(\hat{A}_t, \hat{\varphi}, \hat{\lambda}^{\dot{\alpha}A})$ | vector | – | **anti** | – |
| | $(\hat{a}_{\alpha\dot{\beta}}, \hat{\chi}^A_\alpha)$ | hyper | – | **sym** | – |
| D1′–D5 | $\Lambda_l$ | Fermi | – | **n** | **2N** |
| D1–D1′ | $(\Phi_A, \Psi^{\dot{\alpha}})$ | twisted hyper | **2k** | **n** | – |
| | $\Xi_\alpha$ | Fermi | **2k** | **n** | – |

(b)

Figure 4.5: (a) 1d quiver and (b) matters for $SO(2N)$. (bold/dashed lines for hyper/Fermi)

each Fermi multiplet [47]. Our $(0,4)$ multiplets decompose to $(0,2)$ multiplets




(a)

| Mode | Field | Type | $Sp(k)$ | $O(n)$ | $SO(2N+1)$ |
|---|---|---|---|---|---|
| D1−D1 | $(A_t, \varphi, \lambda^{\dot\alpha A})$ | vector | **sym** | – | – |
| | $(a_{\alpha\dot\beta}, \chi^A_\alpha)$ | hyper | **anti** | – | – |
| D1−D5 | $(q_{\dot\alpha}, \psi^A)$ | hyper | **2k** | – | **2N+1** |
| D1′−D1′ | $(\hat A_t, \hat\varphi, \hat\lambda^{\dot\alpha A})$ | vector | – | **anti** | – |
| | $(\hat a_{\alpha\dot\beta}, \hat\chi^A_\alpha)$ | hyper | – | **sym** | – |
| D1′−D5 | $\Lambda_l$ | Fermi | – | **n** | **2N+1** |
| | $\Lambda$ | Fermi | – | **n** | – |
| D1−D1′ | $(\Phi_A, \Psi^{\dot\alpha})$ | twisted hyper | **2k** | **n** | – |
| | $\Xi_\alpha$ | Fermi | **2k** | **n** | – |

(b)

Figure 4.6: (a) 1d quiver and (b) matters for $SO(2N+1)$.

as follows:

$$\begin{aligned}
\text{vector } (A_t, \varphi, \lambda^{\dot\alpha A}) \quad &\longrightarrow \quad \text{vector } V\ (A_t, \varphi, \lambda^{\dot 1 2}, \lambda^{\dot 2 1}) \ + \ \text{Fermi } \lambda\ (\lambda^{\dot 1 1}, \lambda^{\dot 2 2}) \\
\text{vector } (\hat A_t, \hat\varphi, \hat\lambda^{\dot\alpha A}) \quad &\longrightarrow \quad \text{vector } \hat V\ (\hat A_t, \hat\varphi, \hat\lambda^{\dot 1 2}, \hat\lambda^{\dot 2 1}) \ + \ \text{Fermi } \hat\lambda\ (\hat\lambda^{\dot 1 1}, \hat\lambda^{\dot 2 2}) \\
\text{hyper } (a_{\alpha\dot\beta}, \chi^A_\alpha) \quad &\longrightarrow \quad \text{chiral } B\ (a_{c\dot 1}, \chi^2_c) \ + \ \text{chiral } \tilde B^\dagger\ (a_{c\dot 2}, \chi^1_c) \\
\text{hyper } (\hat a_{\alpha\dot\beta}, \hat\chi^A_\alpha) \quad &\longrightarrow \quad \text{chiral } C\ (\varphi_{c\dot 1}, \xi^2_c) \ + \ \text{chiral } \tilde C^\dagger\ (\varphi_{c\dot 2}, \xi^1_c) \\
\text{hyper } (q_{\dot\alpha}, \psi^A) \quad &\longrightarrow \quad \text{chiral } q\ (q_{\dot 1}, \psi^2) \ + \ \text{chiral } \tilde q^\dagger\ (q_{\dot 2}, \psi^1) \\
\text{twisted hyper } (\Phi_A, \Psi^{\dot\alpha}) \quad &\longrightarrow \quad \text{chiral } \Phi\ (\Phi_1, \Psi^2) \ + \ \text{chiral } \tilde\Phi^\dagger\ (\Phi_2, \Psi^1) \\
\text{Fermi } (\Lambda_l),\ (\Lambda),\ (\Xi_\alpha) \quad &\longrightarrow \quad \text{Fermi } (\Lambda_l),\ (\Lambda),\ (\Xi_\alpha)\ . 
\end{aligned} \tag{4.2.1}$$

The scalars in rank 2 symmetric or antisymmetric representations are real. It decomposes to two $(0,2)$ chiral multiplets whose scalars are complexified as $a_{c\dot\beta} = a_{1\dot\beta} + ia_{2\dot\beta}$ and likewise $\hat{a}_{c\dot\beta}$.

In $(0,2)$ theories, one can turn on two types of holomorphic 'superpotentials' for each Fermi multiplet $\Psi$, $J_\Psi$ and $E_\Psi$. $(0,2)$ supersymmetry demands the superpotentials to satisfy

$$\sum_{\nu\in\text{Fermi}} E_\nu J_\nu = 0 \ . \tag{4.2.2}$$

We first consider the $SO(2N)$ theory, in which case $E$ and $J$ for Fermi multiplets are given by

$$\begin{aligned}
J_\lambda &= \sqrt{2}(q\tilde{q} + [B,\tilde{B}]) & E_\lambda &= -\sqrt{2}\widetilde{\Phi}\Phi \\
J_{\hat\lambda} &= \sqrt{2}[C,\widetilde{C}] & E_{\hat\lambda} &= \sqrt{2}\Phi\widetilde{\Phi} \\
J_{\Xi_1} &= \sqrt{2}(\widetilde{\Phi}\widetilde{C} - \widetilde{B}\widetilde{\Phi}) & E_{\Xi_1} &= \sqrt{2}(C\Phi - \Phi B) \\
J_{\Xi_2} &= -\sqrt{2}(\widetilde{\Phi}C - B\widetilde{\Phi}) & E_{\Xi_2} &= \sqrt{2}(\widetilde{C}\Phi - \Phi\widetilde{B}) \\
J_{\Lambda_l} &= \sqrt{2}\widetilde{q}\widetilde{\Phi} & E_{\Lambda_l} &= \sqrt{2}\Phi q \ .
\end{aligned} \tag{4.2.3}$$

The first two lines, for $(0,4)$ gauginos of $Sp(k)\times O(n)$, are required by demanding $(0,4)$ SUSY enhancement [46]. Namely, gaugino fields' $J$ and $E$ acquire contributions only from hypermultiplets and twisted hypermultiplets, respectively. But with the first two lines only, (4.2.2) is not met. The next three lines are fixed (up to sign choices) by demanding (4.2.2) to hold, as illustrated in [46] in different models. D-terms are given by

$$\begin{aligned}
D_{Sp(k)} &= qq^\dagger - \widetilde{q}^\dagger\widetilde{q} + [B,B^\dagger] - [\widetilde{B}^\dagger,\widetilde{B}] - \Phi^\dagger\Phi + \widetilde{\Phi}\widetilde{\Phi}^\dagger \\
D_{O(n)} &= [C,C^\dagger] - [\widetilde{C}^\dagger,\widetilde{C}] + \Phi\Phi^\dagger - \widetilde{\Phi}^\dagger\widetilde{\Phi}.
\end{aligned} \tag{4.2.4}$$

With these superpotentials and D-terms, the bosonic potential energy is given by [46,47],

$$V = \sum_{G\in\text{gauge}} \frac{1}{2}D_G^2 + \sum_{\nu\in\text{Fermi}} \left(|E_\nu|^2 + |J_\nu|^2\right) . \tag{4.2.5}$$

One can show that (4.2.5) exhibits enhanced $SO(4) = SU(2)_r \times SU(2)_R$ R-symmetry,

$$\begin{aligned} V &= \frac{1}{2}\left(q_{\dot\alpha}(\sigma^m)^{\dot\alpha}{}_{\dot\beta}q^{\dagger\dot\beta} + (\sigma^m)^{\dot\alpha}{}_{\dot\beta}\left[a_{\alpha\dot\alpha}, a^{\dagger\alpha\dot\beta}\right]\right)^2 + \frac{1}{2}\left((\sigma^m)^{\dot\alpha}{}_{\dot\beta}\left[\hat a_{\alpha\dot\alpha}, \hat a^{\dagger\alpha\dot\beta}\right]\right)^2 \\ &+\frac{1}{2}\left(\Phi_A(\sigma^m)^A{}_B\Phi^{\dagger B}\right)^2 + \frac{1}{2}\left(\Phi^{\dagger A}(\bar\sigma^m)_A{}^B\Phi_B\right)^2 + |\Phi_A q_{\dot\alpha}|^2 \\ &+|\hat a_{\alpha\dot\beta}\Phi_A - \Phi_A a_{\alpha\dot\beta}|^2 + |\Phi^{\dagger A}\hat a_{\alpha\dot\beta} - a_{\alpha\dot\beta}\Phi^{\dagger A}|^2. \end{aligned} \tag{4.2.6}$$

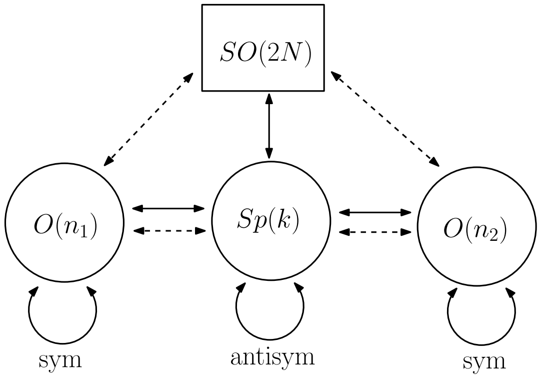
Since $SO(4)$ is the $\mathcal{N} = (0,4)$ R-symmetry, this is a strong indication that the classical action indeed has $(0,4)$ SUSY. We content ourselves with this observation, rather than checking $(0,4)$ SUSY of the full action. The fields in the last expression satisfy the pseudo-reality condition of $Sp(k)$, $\widetilde{q}^T = \Lambda q$, $\widetilde{\Phi}^T = \Phi(\Lambda^{-1})^T$, where $\Lambda$ is the $Sp(k)$ skew-symmetric matrix.

One can repeat the analysis for the $SO(2N+1)$ quiver. One point to note here is that there is no superpotential for the Fermi multiplet $\Lambda$. So despite the presence of $2N+2$ $O(n)$ fundamental Fermi multiplets $\Lambda_l$, $\Lambda$, their flavor symmetry is $SO(2N+1)$, as we expect from 5d bulk.

When there are two 5d hypermultiplets in the spinor representation of $SO(N)$, we can consider a sector with $n_1$ and $n_2$ particles and $k$ instantons. The 1d quivers and fields are shown in Figs. 4.7, 4.8. The Lagrangians can be constructed by following the completely same procedures, which we do not present here.

### 4.2.2 The instanton partition functions

We shall compute the Witten indices of the quantum mechanics presented in the previous subsection. They count BPS states preserving $Q = -\bar Q^{1\dot 2}$ and




(a)

| Mode | Field | Type | $Sp(k)$ | $O(n_1)$ | $O(n_2)$ | $SO(2N)$ |
|---|---|---|---|---|---|---|
| D1−D1 | $(A_t, \varphi, \lambda^{\dot{\alpha}A})$ | vector | **sym** | − | − | − |
| | $(a_{\alpha\dot{\beta}}, \chi^A_\alpha)$ | hyper | **anti** | − | − | − |
| D1−D5 | $(q_{\dot{\alpha}}, \psi^A)$ | hyper | **2k** | − | − | **2N** |
| D1′−D1′ | $(\hat{A}_t, \hat{\varphi}, \hat{\lambda}^{\dot{\alpha}A})$ | vector | − | **anti** | − | − |
| | $(\hat{a}_{\alpha\dot{\beta}}, \hat{\chi}^A_\alpha)$ | hyper | − | **sym** | − | − |
| D1′−D5 | $\Lambda_l$ | Fermi | − | $\boldsymbol{n_1}$ | − | **2N** |
| D1−D1′ | $(\Phi_A, \Psi^{\dot{\alpha}})$ | twisted hyper | **2k** | $\boldsymbol{n_1}$ | − | − |
| | $\Xi_\alpha$ | Fermi | **2k** | $\boldsymbol{n_1}$ | − | − |
| D1″−D1″ | $(\hat{A}_t, \hat{\varphi}, \hat{\lambda}^{\dot{\alpha}A})$ | vector | − | − | **anti** | − |
| | $(\hat{a}_{\alpha\dot{\beta}}, \hat{\chi}^A_\alpha)$ | hyper | − | − | **sym** | − |
| D1″−D5 | $\Lambda_l$ | Fermi | − | − | $\boldsymbol{n_2}$ | **2N** |
| D1−D1″ | $(\Phi_A, \Psi^{\dot{\alpha}})$ | twisted hyper | **2k** | − | $\boldsymbol{n_2}$ | − |
| | $\Xi_\alpha$ | Fermi | **2k** | − | $\boldsymbol{n_2}$ | − |

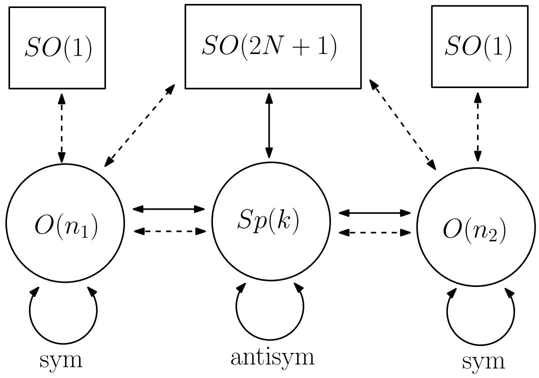
(b)

Figure 4.7: The 1d quiver (a) and matters (b) for 5d $SO(2N)$ theory with two hypermultiplets.

$Q^\dagger = \bar{Q}^{2\dot{1}}$, and is defined by

$$Z_{QM} = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-2\epsilon_+(J_r+J_R)} e^{-2\epsilon_- J_l} e^{-v_i q_i} e^{-z\cdot F}\right]. \tag{4.2.7}$$

$J_l$, $J_r$ and $J_R$ are Cartans of $SO(4) = SU(2)_l \times SU(2)_r$ and $SU(2)_R$ respectively, while $q_i$ are the $SO(N)$ electric charges. $F$, $z$ denote other charges and their chemical potentials.




(a)

| Mode | Field | Type | $Sp(k)$ | $O(n_1)$ | $O(n_2)$ | $SO(2N+1)$ |
|---|---|---|---|---|---|---|
| D1−D1 | $(A_t, \varphi, \lambda^{\dot{\alpha}A})$ | vector | **sym** | − | − | − |
| | $(a_{\alpha\dot{\beta}}, \chi^A_\alpha)$ | hyper | **anti** | − | − | − |
| D1−D5 | $(q_{\dot{\alpha}}, \psi^A)$ | hyper | **2k** | − | − | **2N+1** |
| D1′−D1′ | $(\hat{A}_t, \hat{\varphi}, \hat{\lambda}^{\dot{\alpha}A})$ | vector | − | **anti** | − | − |
| | $(\hat{a}_{\alpha\dot{\beta}}, \hat{\chi}^A_\alpha)$ | hyper | − | **sym** | − | − |
| D1′−D5 | $\Lambda_l$ | Fermi | − | $\boldsymbol{n_1}$ | − | **2N+1** |
| | $\Lambda$ | Fermi | − | $\boldsymbol{n_1}$ | − | − |
| D1−D1′ | $(\Phi_A, \Psi^{\dot{\alpha}})$ | twisted hyper | **2k** | $\boldsymbol{n_1}$ | − | − |
| | $\Xi_\alpha$ | Fermi | **2k** | $\boldsymbol{n_1}$ | − | − |
| D1″−D1″ | $(\hat{A}'_t, \hat{\varphi}', \hat{\lambda}'^{\dot{\alpha}A})$ | vector | − | − | **anti** | − |
| | $(\hat{a}'_{\alpha\dot{\beta}}, \hat{\chi}'^A_\alpha)$ | hyper | − | − | **sym** | − |
| D1″−D5 | $\Lambda'_l$ | Fermi | − | − | $\boldsymbol{n_2}$ | **2N+1** |
| | $\Lambda'$ | Fermi | − | − | $\boldsymbol{n_2}$ | − |
| D1−D1″ | $(\Phi'_A, \Psi'^{\dot{\alpha}})$ | twisted hyper | **2k** | − | $\boldsymbol{n_2}$ | − |
| | $\Xi'_\alpha$ | Fermi | **2k** | − | $\boldsymbol{n_2}$ | − |

(b)

Figure 4.8: The 1d quiver (a) and matters (b) for 5d $SO(2N+1)$ theory with two hypermultiplets.

We compute (4.2.7) using the contour integral formula of [12, 22, 26]. The zero modes in the path integral appear as the contour integral variables. They are the eigenvalues of the scalar $\varphi$ and $A_\tau$ in the vector multiplet. For $O(n)$, the flat connections on $S^1$ have two disconnected sectors $O(n)_\pm$. $U = e^{R\phi} \equiv$

$e^{R(\varphi+iA_\tau)}$, where $R$ is the radius of the temporal circle, is given by

$$\begin{aligned}
&U^+_{O(2n)} = \text{diag}(e^{i\phi_1\sigma_2}, e^{i\phi_2\sigma_2}, \cdots, e^{i\phi_n\sigma_2}) \ \ , \ \ U^-_{O(2n)} = \text{diag}(e^{i\phi_1\sigma_2}, \cdots, e^{i\phi_{n-1}\sigma_2}, \sigma_3) \\
&U^+_{O(2n+1)} = \text{diag}(e^{i\phi_1\sigma_2}, \cdots, e^{i\phi_n\sigma_2}, 1) \ \ , \ \ U^-_{O(2n+1)} = \text{diag}(e^{i\phi_1\sigma_2}, \cdots, e^{i\phi_n\sigma_2}, -1) \\
&U_{Sp(k)} = \text{diag}(e^{i\phi_1\sigma_2}, \cdots, e^{i\phi_k\sigma_2}) \ . \qquad (4.2.8)
\end{aligned}$$

$\sigma_i$ are Pauli matrices, 'diag' mean block-diagonal matrices, and $\det(U^\pm) = \pm 1$. The integrand acquires contributions from various multiplets. A chiral multiplet $\Phi$ and a Fermi multiplet $\Psi$ contribute as

$$Z_\Phi = \prod_{\rho\in R_\Phi} \frac{1}{2\sinh(\frac{\rho(\phi)+2J\epsilon_+ + Fz}{2})} \ \ , \ \ Z_\Psi = \prod_{\rho\in R_\Psi} 2\sinh(\frac{\rho(\phi)+2J\epsilon_+ + Fz}{2}) \qquad (4.2.9)$$

respectively. $\rho$ runs over the weights of $Sp(k)$, $O(n)$ in the representation $R_\Phi$, $R_\Psi$, and $J$ is defined by $J = J_r + J_R$. $F$ collectively denotes the remaining charges. A $(0,2)$ vector multiplet $V$ contributes similarly as $Z_V = \prod_{\alpha\in\text{root}} 2\sinh\frac{\alpha(\phi)}{2}$, where we used the formula for a Fermi multiplet at $J = 0$, $F = 0$. Collecting all, the Witten index is given by

$$Z = \frac{1}{|W|}\oint \frac{d\phi}{2\pi i} Z_{\text{1-loop}} \ \ , \quad Z_{\text{1-loop}} \equiv \prod_V Z_V \prod_\Phi Z_\Phi \prod_\Psi Z_\Psi \ . \qquad (4.2.10)$$

The $O(n)$ holonomy has two discrete sectors. The Witten index is given by [49]

$$Z = \frac{Z^+ + Z^-}{2} \ . \qquad (4.2.11)$$

The Weyl factors $|W|$ of $O(2n)_\pm$, $O(2n+1)_\pm$, $Sp(k)$ are given by [49]

$$\begin{aligned}
|W_{O(2n)_+}| = \frac{1}{2^{n-1}n!} \ , \ |W_{O(2n)_-}| = \frac{1}{2^{n-1}(n-1)!} \ , \\
|W_{O(2n+1)_+}| = |W_{O(2n+1)_-}| = \frac{1}{2^n n!} \ , |W_{Sp(k)}| = \frac{1}{2^k k!} \ . \qquad (4.2.12)
\end{aligned}$$

For $SO(N)$ with odd $N$, one can show that $Z_{\text{1-loop}} = 0$ in $O(2n)_-$ and $O(2n+1)_+$ sectors, since the fermionic zero modes from $\Lambda$ (in Table 4.6(b)) provide factors of 0's.

Let us call $Z_{k,n}$ the index of the $Sp(k)\times O(n)$ quiver. Being a multi-particle index, it acquires contribution from $n$ hypermultiplet particles either bound or unbound to $k$ instantons. Also, as we shall explain in more detail below, $Z_{k,n}$ for $n\geq 2$ also contains a spurious contribution from particles not belonging to the 5d QFT. To explain these structures clearly, we first discuss the indices $Z_{0,n}$ before considering the instanton partition functions at $k\neq 0$. At $n=1$, $k=0$, the $O(1)$ indices do not contain integrals. The results are given by

$$\begin{aligned}
Z_{0,1}^{SO(2N)} &= \frac{1}{2}\left(\frac{\prod_{l=1}^{N} 2\sinh\frac{v_l}{2}}{2\sinh\frac{\epsilon_1}{2}\cdot 2\sinh\frac{\epsilon_2}{2}}+\frac{\prod_{l=1}^{N} 2\cosh\frac{v_l}{2}}{2\sinh\frac{\epsilon_1}{2}\cdot 2\sinh\frac{\epsilon_2}{2}}\right)\\
Z_{0,1}^{SO(2N+1)} &= \frac{\prod_{l=1}^{N} 2\cosh\frac{v_l}{2}}{2\sinh\frac{\epsilon_1}{2}\cdot 2\sinh\frac{\epsilon_2}{2}}\ .
\end{aligned}\tag{4.2.13}$$

The overall factors $(2\sinh\frac{\epsilon_1}{2}\cdot 2\sinh\frac{\epsilon_2}{2})^{-1}$ in (4.2.13) come from the center-of-mass motion on $\mathbb{R}^4$. The remaining factor is the character of the $SO(2N)$ chiral spinor $\frac{\prod_{l=1}^{N} 2\sinh\frac{v_l}{2}+\prod_{l=1}^{N} 2\cosh\frac{v_l}{2}}{2}$, and that of $SO(2N+1)$ spinor $\prod_{l=1}^{N} 2\cosh\frac{v_l}{2}$, respectively. They are the perturbative partition functions of matters in $SO(N)$ in spinor representations. Next, $Z_{0,2}$ is given by

$$\begin{aligned}
Z_{0,2}^{SO(2N)} &= \frac{1}{2}\left[\oint\frac{d\chi}{2\pi i}\frac{2\sinh\epsilon_+\cdot\prod_{l=1}^{N} 2\sinh(\frac{v_l\pm\chi}{2})}{2\sinh(\frac{\epsilon_{1,2}}{2})\cdot 2\sinh(\frac{\epsilon_{1,2}\pm 2\chi}{2})}+\frac{2\cosh\epsilon_+\cdot\prod_{l=1}^{N} 2\sinh v_l}{2\cosh\frac{\epsilon_{1,2}}{2}\cdot(2\sinh\frac{\epsilon_{1,2}}{2})^2}\right]\\
Z_{0,2}^{SO(2N+1)} &= \frac{1}{2}\oint\frac{d\chi}{2\pi i}\frac{2\sinh\epsilon_+\cdot\prod_{l=1}^{N} 2\sinh(\frac{v_l\pm\chi}{2})\cdot 2\sinh(\pm\frac{\chi}{2})}{2\sinh(\frac{\epsilon_{1,2}}{2})\cdot 2\sinh(\frac{\epsilon_{1,2}\pm 2\chi}{2})}.
\end{aligned}\tag{4.2.14}$$

For $SO(2N+1)$ and the first term of $SO(2N)$ index, one should evaluate JK-Res. With the choice $\eta>0$, one keeps the residues at $\chi=-\frac{\epsilon_{1,2}}{2}$ and $\chi=-\frac{\epsilon_{1,2}}{2}+\pi i$.[6] For $N\leq 6$, one obtains

$$Z_{0,2}^{SO(N)}=\frac{Z_{0,1}^{SO(N)}(\epsilon_\pm,v_l)^2+Z_{0,1}^{SO(N)}(2\epsilon_\pm,2v_l)}{2}\ ,\tag{4.2.15}$$

[6]For $O(n)$ and $Sp(k)$ gauge theories, the choice of $\eta$ does not affect the results due to Weyl symmetry [22].

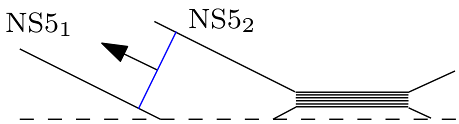


Figure 4.9: D1′-brane escaping the 5d QFT

while for $N = 7, 8$ one obtains

$$Z_{0,2}^{SO(N)} = \frac{Z_{0,1}^{SO(N)}(\epsilon_\pm, v_l)^2 + Z_{0,1}^{SO(N)}(2\epsilon_\pm, 2v_l)}{2} - \frac{1}{2} \cdot \frac{2\cosh \epsilon_+}{2\sinh \frac{\epsilon_{1,2}}{2}} \ . \tag{4.2.16}$$

(4.2.15) and the first term of (4.2.16) are the indices of two non-interacting identical particles, whose single particle index is given by $Z_{0,1}^{SO(N)}$. There are no bound states formed by these perturbative hypermultiplet particles, as expected. The second term of (4.2.16) requires more explanations, which we now turn to.

The second term of (4.2.16) comes from extra states in the brane system that do not belong to the 5d QFT. In particular, the fractional coefficient in the fugacity expansion implies that it comes from a sector which has a continuum unlifted by our massive deformations. In fact, following the arguments presented between (4.1.28) and (4.1.29), one finds that the linear 1-loop potential from (4.2.14) vanishes for $N = 7, 8$, implying continua. Physically, this comes from a D1′-brane moving away from 5d QFT, suspended between two parallel 5-branes as in Fig. 4.9. Although we are not aware of fully logical arguments, it has been empirically observed that the last term $-\frac{1}{2}\frac{2\cosh\epsilon_+}{2\sinh\frac{\epsilon_{1,2}}{2}}$ is the contribution from the escaping particle for strings suspended between parallel 5-branes: e.g. see eqn.(3.62) of [22]. See also [50–52] for related results. The suspended string of Fig. 4.9 carries the same spacetime and R-symmetry quantum numbers as a 5d vector multiplet particles, since the configuration of Fig. 4.9 is locally dual to a fundamental string suspended between two D5-branes (a 5d vector W-boson).

Indeed, the chemical potential dependence $\sim \frac{2\cosh\epsilon_+}{2\sinh\frac{\epsilon_{1,2}}{2}}$ is precisely that of a 5d W-boson and its superpartners. Such extra states start to appear at $n \geq 2$, since at $n = 1$, one only has fractional D1$'$ stuck to O5.

Collecting all, we expect that the partition function at $k = 0$ is given by

$$\sum_{n=0}^{\infty} e^{-nm} Z_{0,n}^{SO(N)} = Z_{\text{pert}} \equiv \text{PE}\left[e^{-m} Z_{0,1}^{SO(N)}\right] \tag{4.2.17}$$

for $N \leq 6$, while for $N = 7, 8$ we expect that it is given by

$$\sum_{n=0}^{\infty} e^{-nm} Z_{0,n}^{SO(N)} = Z_{\text{pert}} Z_{\text{extra}} = \text{PE}\left[e^{-m} Z_{0,1}^{SO(N)}\right] \text{PE}\left[-\frac{1}{2} e^{-2m} \frac{2\cosh\epsilon_+}{2\sinh\frac{\epsilon_{1,2}}{2}}\right] . \tag{4.2.18}$$

Here, $Z_{0,0} \equiv 1$ by definition, and $PE\left[f(x, y, \cdots)\right] \equiv \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(nx, ny, \cdots)\right]$ is the multi-particle index for the single particle index $f$.

The full partition function would factorize as

$$\sum_{k,n=0}^{\infty} q^k e^{-nm} Z_{k,n}^{SO(N)} = Z_{\text{inst}}(q, \epsilon_{1,2}, m) Z_{\text{pert}}(\epsilon_{1,2}, m) Z_{\text{extra}}(\epsilon_{1,2}, m). \tag{4.2.19}$$

Expanding $Z_{\text{inst}}(q) = \sum_{k=0}^{\infty} Z_k q^k$ where $Z_0 \equiv 1$, (4.2.19) implies at given $q^k$ order that

$$\sum_{n=0}^{\infty} e^{-nm} Z_{k,n} = Z_k Z_{\text{pert}} Z_{\text{extra}}. \tag{4.2.20}$$

When there are two 5d hypermultiplets as in Fig. 4.4, the full partition function is

$$Z^{(2)} = \sum_{k,n_1,n_2=0}^{\infty} q^k e^{-n_1 m_1 - n_2 m_2} Z_{k_1,n_1,n_2} \tag{4.2.21}$$

where $Z_{k,n_1,n_2}$ is the index for $k$ D1, $n_1$ D1$'$ and $n_2$ D1$''$. $m_{1,2}$ are the $Sp(2)$ flavor chemical potentials. The contributions from perturbative and extra degrees of freedom in this case are

$$
\begin{aligned}
Z^{(2)}_{\text{pert}}(\epsilon_{1,2}, m_{1,2}) &= Z_{\text{pert}}(\epsilon_{1,2}, m_1) Z_{\text{pert}}(\epsilon_{1,2}, m_2) \\
Z^{(2)}_{\text{extra}}(\epsilon_{1,2}, m_{1,2}) &= Z_{\text{extra}}(\epsilon_{1,2}, m_1) Z_{\text{extra}}(\epsilon_{1,2}, m_2) \qquad (4.2.22)
\end{aligned}
$$

where $Z_{\text{pert}}$ and $Z_{\text{extra}}$ take the same forms as in (4.2.18).

Although our methods apply well to both $SO(8)$ and $SO(7)$, we only study the cases with $SO(7)$ in this paper. We start from the case with one hypermultiplet field. From the field contents of Fig. 4.6, $Z_{\text{1-loop}}$ for $k$ instantons and $2n$(=even) hypermultiplet particles is given by

$$
\begin{aligned}
[Z_{\text{1-loop}}]^{SO(7)}_{k,2n} &= \frac{1}{2^k\, k!} \frac{(2\sinh\epsilon_+)^k \cdot \prod_{i=1}^k 2\sinh(\epsilon_+ \pm \phi_i)\, 2\sinh(\pm\phi_i) \cdot \prod_{i>j} 2\sinh(\frac{2\epsilon_+ \pm\phi_i\pm\phi_j}{2})\, 2\sin\cdots}{(2\sinh(\frac{\epsilon_{1,2}}{2}))^k \cdot \prod_{i>j} 2\sinh(\frac{\epsilon_{1,2}\pm\phi_i\pm\phi_j}{2}) \cdot \prod_{i=1}^k \prod_{l=1}^3 2\sinh(\frac{\epsilon_+\pm\phi_i\pm v_l}{2}) \cdot \prod_{i=1}^k 2\,\mathrm{s}\cdots} \\
&\cdot \frac{1}{2^n\, n!} \frac{(2\sinh\epsilon_+)^n \cdot \prod_{I>J} 2\sinh(\frac{2\epsilon_+\pm\chi_I\pm\chi_J}{2}) \cdot \prod_{I>J} 2\sinh(\frac{\pm\chi_I\pm\chi_J}{2})}{(2\sinh(\frac{\epsilon_{1,2}}{2}))^n \cdot \prod_{I=1}^n 2\sinh(\frac{\epsilon_{1,2}\pm 2\chi_I}{2}) \cdot \prod_{I>J} 2\sinh(\frac{\epsilon_{1,2}\pm\chi_I\pm\chi_J}{2})} \\
&\cdot \prod_{I=1}^n \prod_{l=1}^3 2\sinh(\frac{\pm\chi_I + v_l}{2}) \cdot \prod_{I=1}^n 2\sinh(\frac{\chi_I}{2}) \cdot \frac{\prod_{i=1}^k \prod_{I=1}^n 2\sinh(\frac{\epsilon_-\pm\phi_i\pm\chi_I}{2})}{\prod_{i=1}^k \prod_{I=1}^n 2\sinh(\frac{-\epsilon_+\pm\phi_i\pm\chi_I}{2})}\,,
\end{aligned}
$$

while $Z_{\text{1-loop}}$ for $k$ instantons and $2n+1$ hypermultiplet particles is given by

$$
\begin{aligned}
[Z_{\text{1-loop}}]^{SO(7)}_{k,2n+1} &= \frac{1}{2^k\, k!} \frac{(2\sinh\epsilon_+)^k \cdot \prod_{i=1}^k 2\sinh(\epsilon_+ \pm \phi_i)\, 2\sinh(\pm\phi_i) \cdot \prod_{i>j} 2\sinh(\frac{2\epsilon_+ \pm\phi_i\pm\phi_j}{2})\, 2\sin\cdots}{(2\sinh(\frac{\epsilon_{1,2}}{2}))^k \cdot \prod_{i>j} 2\sinh(\frac{\epsilon_{1,2}\pm\phi_i\pm\phi_j}{2}) \cdot \prod_{i=1}^k \prod_{l=1}^3 2\sinh(\frac{\epsilon_+\pm\phi_i\pm v_l}{2}) \cdot \prod_{i=1}^k 2\cdots} \\
&\cdot \frac{1}{2^n\, n!} \frac{(2\sinh\epsilon_+)^n \cdot \prod_{I>J} 2\sinh(\frac{2\epsilon_+\pm\chi_I\pm\chi_J}{2}) \cdot \prod_{I>J} 2\sinh(\frac{\pm\chi_I\pm\chi_J}{2})}{(2\sinh(\frac{\epsilon_{1,2}}{2}))^{n+1} \cdot \prod_{I=1}^n 2\sinh(\frac{\epsilon_{1,2}\pm 2\chi_I}{2}) \cdot \prod_{I>J} 2\sinh(\frac{\epsilon_{1,2}\pm\chi_I\pm\chi_J}{2}) \prod_{I=1}^n 2\,\mathrm{co}\cdots} \\
&\cdot \prod_{I=1}^n \prod_{l=1}^3 2\sinh(\frac{\pm\chi_I + v_l}{2}) \cdot \prod_{I=1}^n 2\sinh(\frac{\chi_I}{2}) \cdot \prod_{I=1}^n 2\cosh(\frac{2\epsilon_+ \pm \chi_I}{2}) \cdot \prod_{I=1}^n 2\cosh(\frac{\chi_I}{2}) \cdots \\
&\cdot \frac{\prod_{i=1}^k \prod_{I=1}^n 2\sinh(\frac{\epsilon_-\pm\phi_i\pm\chi_I}{2}) \cdot \prod_{i=1}^k 2\cosh(\frac{\epsilon_-\pm\phi_i}{2})}{\prod_{i=1}^k \prod_{I=1}^n 2\sinh(\frac{-\epsilon_+\pm\phi_i\pm\chi_I}{2}) \cdot \prod_{i=1}^k 2\cosh(\frac{-\epsilon_+\pm\phi_i}{2})}\,.
\end{aligned}
$$

$i, j = 1, \cdots, k$ are $Sp(k)$ indices, $I, J = 1, \cdots, n$ are $O(2n)$ or $O(2n+1)$ indices, and $l = 1, 2, 3$ are $SO(7)$ indices. (4.2.23) and (4.2.24) are computed on either $O(n)_+$ or $O(n)_-$ sector, where $\chi_I$ are eigenvalues of $\log U^{\pm}$ given by (4.2.8).

The partition function at $k=1$, $n=0$ is given by

$$Z_{1,0}^{SO(7)} = \oint \frac{d\phi}{2\pi i}\, \frac{1}{2} \cdot \frac{2\sinh \epsilon_+ \cdot 2\sinh(\epsilon_+ \pm \phi) \cdot 2\sinh(\pm\phi)}{2\sinh(\frac{\epsilon_{1,2}}{2}) \cdot \prod_{l=1}^{3} 2\sinh(\frac{\epsilon_+ \pm \phi \pm v_l}{2}) \cdot 2\sinh(\frac{\epsilon_+ \pm \phi}{2})}\,. \tag{4.2.25}$$

Poles chosen at $\eta > 0$ are $\phi = -\epsilon_+$, $\phi = -\epsilon_+ \pm v_l$, but the residue from $\phi = -\epsilon_+$ vanishes. Collecting the residues, one obtains

$$\begin{aligned} Z_{1,0}^{SO(7)} &= \frac{t}{(1-tu)(1-t/u)} \prod_{i<j} \frac{t^4}{(1-t^2 b_i^{\pm} b_j^{\pm})} (\chi_{\mathbf{9}}^{SU(2)} + \chi_{\mathbf{7}}^{SU(2)} (\chi_{\mathbf{7}}^{SO(7)} + 1) \\ &\quad + \chi_{\mathbf{5}}^{SU(2)} (-\chi_{\mathbf{35}}^{SO(7)} + \chi_{\mathbf{7}}^{SO(7)} + 1) + \chi_{\mathbf{3}}^{SU(2)} (-\chi_{\mathbf{35}}^{SO(7)} + \chi_{\mathbf{27}} + 1) \\ &\quad + \chi_{\mathbf{105}}^{SO(7)} - \chi_{\mathbf{21}}^{SO(7)} + \chi_{\mathbf{7}}^{SO(7)}) \\ &= \frac{t}{(1-tu)(1-t/u)} \sum_{p=0}^{\infty} \chi_{(0,p,0)}(v_l)\, t^{2p+4} \end{aligned} \tag{4.2.26}$$

where $t = e^{-\epsilon_+}$ and $u = e^{-\epsilon_-}$. Here $\chi_{\mathbf{R}}$ is the character of $SO(7)$ representation $\mathbf{R}$. This is simply the well-known 1-instanton partition function of $SO(7)$ gauge theory. E.g. see [53] for the above character expansion form.

Next, consider the sector at $k=1$, $n=1$. $Z_{1,1}^{SO(7)}$ is given by

$$Z_{1,1}^{SO(7)} = \oint \frac{d\phi}{2\pi i} \left[Z_{\text{1-loop}}\right]_{1,0}^{SO(7)} \frac{\prod_{l=1}^{3} 2\cosh(\frac{v_l}{2})}{2\sinh(\frac{\epsilon_{1,2}}{2})} \cdot \frac{2\cosh(\frac{\epsilon_- \pm \phi}{2})}{2\cosh(\frac{-\epsilon_+ \pm \phi}{2})}. \tag{4.2.27}$$

Poles chosen at $\eta > 0$ with nonzero residues are at $\phi = -\epsilon_+ \pm v_l$. As we explained around (4.2.20), $Z_{1,1}^{SO(7)}$ has contributions from $Z_{\text{pert}}$ at $n=1$. Let us call the proper contribution to the instanton partition function $\hat{Z}_{k,n}^{SO(7)}$. From (4.2.20), one obtains

$$\hat{Z}_{1,1}^{SO(7)} = Z_{1,1}^{SO(7)} - Z_{1,0}^{SO(7)} Z_{0,1}^{SO(7)}\,. \tag{4.2.28}$$

Hat denotes the instanton partition function at level $(k,n)$, while $Z_{k,n}$ is simply the Witten index of our $Sp(k) \times O(n)$ quantum mechanics. From this formula, one obtains

$$\begin{aligned}\hat{Z}^{SO(7)}_{1,1} &= \frac{t}{(1-tu^{\pm 1})}\prod_{i<j}\frac{t^4}{(1-t^2b_i^{\pm}b_j^{\pm})}(-\chi^{SU(2)}_{\mathbf{8}}\chi^{SO(7)}_{\mathbf{8}}-\chi^{SU(2)}_{\mathbf{6}}\chi^{SU(2)}_{\mathbf{8}}+\chi^{SU(2)}_{\mathbf{4}}\chi^{SO(7)}_{\mathbf{112}} \\ &\quad -\chi^{SU(2)}_{\mathbf{2}}\chi^{SO(7)}_{\mathbf{168}}) \\ &= -\frac{t}{(1-tu^{\pm 1})}\sum_{p=0}^{\infty}\chi_{(0,p,1)}\,t^{2p+5} \end{aligned} \tag{4.2.29}$$

where $\sum_{p=0}^{\infty}\chi_{(0,p,1)}\,t^{2p+5}=\chi_{\mathbf{8}}(v_l)+\chi_{\mathbf{112}}(v_l)\,t^2+\chi_{\mathbf{720}}(v_l)\,t^4+\cdots$. Then consider the sector at $k=1$, $n=2$. $Z^{SO(7)}_{1,2}$ is given by the $Sp(1)\times O(2)$ contour integral

$$Z^{SO(7)}_{1,2}=\oint\frac{d\phi\,d\chi}{(2\pi i)^2}\,[Z_{\text{1-loop}}]^{SO(7)}_{1,0}\cdot\frac{1}{2}\cdot\frac{2\sinh\epsilon_+\cdot\prod_{l=1}^{3}2\sinh(\frac{\pm\chi+v_l}{2})\cdot 2\sinh(\frac{\pm\chi}{2})}{2\sinh(\frac{\epsilon_{1,2}}{2})\cdot 2\sinh(\frac{\epsilon_{1,2}\pm 2\chi}{2})}\cdot\frac{2\sinh(\frac{\epsilon_-\pm\phi\pm\chi}{2})}{2\sinh(\frac{-\epsilon_+\pm\phi\pm\chi}{2})} \tag{4.2.30}$$

Taking $\eta=(1,1+\epsilon)$ for small positive $\epsilon$ [22], the poles at $(\phi,\chi)=(-\epsilon_+\pm v_l,-\frac{\epsilon_{1,2}}{2}[+\pi i]),(-\epsilon_+\pm v_l,\pm v_l),(\epsilon_+\pm v_l,\mp v_l),(0[+\pi i],\epsilon_++[\pi i]),(\frac{\epsilon_{1,2}}{2}[+\pi i],-\frac{\epsilon_{2,1}}{2}[+\pi i]),(\frac{3\epsilon_+\pm\epsilon_-}{2},-\frac{\epsilon_{1,2}}{2}[+\pi i])$ are chosen. $[+\pi i]$ means that there are two cases with and without $+\pi i$ addition. Subtracting the contribution from $Z_{\text{pert}}Z_{\text{extra}}$ in (4.2.20), the instanton partition function $\hat{Z}^{SO(7)}_{1,2}$ at this order is given by $\hat{Z}_{1,2}=Z_{1,2}-\hat{Z}_{1,1}Z_{0,1}-Z_{1,0}Z_{0,2}$. One finds after computations that

$$\hat{Z}^{SO(7)}_{1,2}=Z^{SO(7)}_{1,0}. \tag{4.2.31}$$

For $n\geq 3$, we find $\hat{Z}_{1,n}=0$. We checked this exactly for $n=3$. For $n=4$, to save time, we plugged in random numbers in the chemical potentials and checked that $\hat{Z}^{SO(7)}_{1,4}$ is very small. (Below, we present an argument for this phenomenon.)

Collecting all the computations at $n=0,1,2$, one obtains

$$
\begin{aligned}
Z_{k=1} &= e^{m}\left[Z_{1,0}+e^{-m}\hat{Z}_{1,1}+e^{-2m}\hat{Z}_{1,2}\right] \\
&= \frac{t}{(1-tu^{\pm 1})}\prod_{i<j}\frac{t^4}{(1-t^2 b_i^{\pm}b_j^{\pm})}\Big[-\chi_{\mathbf{8}}^{SU(2)}\chi_{\mathbf{8}}^{SO(7)}-\chi_{\mathbf{6}}^{SU(2)}\chi_{\mathbf{8}}^{SO(7)}+\chi_{\mathbf{4}}^{SU(2)}\chi_{\mathbf{112}}^{SO(7)}-\chi_{\mathbf{2}}^{SU(2)} \\
&\quad +\Big(\chi_{\mathbf{9}}^{SU(2)}+\chi_{\mathbf{7}}^{SU(2)}(\chi_{\mathbf{7}}^{SO(7)}+1)+\chi_{\mathbf{5}}^{SU(2)}(-\chi_{\mathbf{35}}^{SO(7)}+\chi_{\mathbf{7}}^{SO(7)}+1) \\
&\quad +\chi_{\mathbf{3}}^{SU(2)}(-\chi_{\mathbf{35}}^{SO(7)}+\chi_{\mathbf{27}}+1)+\chi_{\mathbf{105}}^{SO(7)}-\chi_{\mathbf{21}}^{SO(7)}+\chi_{\mathbf{7}}^{SO(7)}\Big)(e^{m}+e^{-m})\Big]
\end{aligned}
$$

Here we multiplied an overall factor $e^m$, like the 'zero point energy' factor, to have the expected Weyl symmetry $m\rightarrow -m$ of the $Sp(1)$ flavor symmetry. Noting that $e^m+e^{-m}=\chi_{\mathbf{2}}^{Sp(1)}$, (4.2.32) completely agrees with (4.1.30), supporting our ADHM-like proposals of section 2 at $n_{\mathbf{8}}=1$.

Here we discuss more about the maximal value of $n$ with $\hat{Z}_{k,n}^{SO(7)}\neq 0$, at given $k$. Note that

$$
Z_{\text{inst}}(q,\epsilon_{1,2},v,m)=e^{-\epsilon_0}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}q^k e^{-nm}\hat{Z}_{k,n}^{SO(7)}(\epsilon_{1,2},v)\ , \tag{4.2.33}
$$

refining the previous definition by the zero point energy-like factor. Note that $m$ is the flavor chemical potential for the 5d hypermultiplet. Since a hypermultiplet only adds fermion zero modes on the instanton moduli space, the rotation parameter $m$ acts only on these fermions. So unlike the chemical potentials $v_i$, $\epsilon_{1,2}$ which act on noncompact zero modes, the coefficient $Z_k$ of $Z_{\text{inst}}$ at given $q^k$ order should not have any poles in $m$. Since $Z_k$ admits fugacity expansions, this implies that $Z_k$ is a finite polynomial in $e^m$ and $e^{-m}$. So the sum over $n$ should truncate to $0\leq n\leq n_{\max}$ for some finite $n_{\max}$, also with a suitable $m$ dependent $\epsilon_0$ to ensure the Weyl symmetry of $Sp(1)$. One can also naturally infer the value of $n_{\max}$. To see this, note that a 5d hypermultiplet in the spinor representation induces $kD(\mathbf{8})=2kT(\mathbf{8})=2k$ complex fermion zero modes on the moduli space, where we used $2T=2^{N-2}$ for $SO(2N+1)$ spinor representation. Quantizing them into $2k$ pairs of fermionic harmonic oscillators, each oscillator raises/lowers the particle number $n$ by 1. This means that the charge difference between the lowest and highest states is $2k$, implying $n_{\rm max} = 2k$. Then $Sp(1)$ Weyl symmetry implies $n \to -n$ symmetry, demanding $\epsilon_0 = -km$ and $\hat{Z}_{k,2k-n} = \hat{Z}_{k,n}$. These completely agree with our empirical findings around (4.2.31). Below, we shall proceed with these properties assumed.

One can study the case with $k = 2$ in the same manner. We computed it at $v_l = \epsilon_- = 0$ due to computational complications. We simply report the results:

$$\begin{aligned}
Z_{2,0}^{SO(7)} &= \frac{t^{10}}{(1-t)^{20}(1+t)^{10}(1+t+t^2)^9}(1+t+15t^2+48t^3+152t^4+446t^5+1126t^6+2374t^7 \\
&\quad +4674t^8+8184t^9+12680t^{10}+17816t^{11}+22957t^{12}+26449t^{13}+27622t^{14}+\cdots+t^{28}) \\
\hat{Z}_{2,1}^{SO(7)} &= -\frac{8\,t^{11}}{(1-t)^{20}(1+t)^{10}(1+t+t^2)^9}(1+3t+17t^2+62t^3+183t^4+477t^5+1109t^6+220 \\
&\quad +3921t^8+6285t^9+9004t^{10}+11543t^{11}+13459t^{12}+14194t^{13}+\cdots+t^{26}) \\
\hat{Z}_{2,2}^{SO(7)} &= \frac{t^{10}}{(1-t)^{20}(1+t)^{10}(1+t+t^2)^9}(1+3t+45t^2+176t^3+647t^4+2087t^5+5560t^6+126 \\
&\quad +25923t^8+46880t^9+74843t^{10}+107589t^{11}+139877t^{12}+162758t^{13}+170752t^{14}+\cdots+
\end{aligned} \tag{4.2.34}$$

Here, the omitted terms in $\cdots$ can be restored from the fact that coefficients of $t^p$ and $t^{28-p}$ are same in the numerator of $Z_{2,0}$, and also from similar reflection symmetries in $\hat{Z}_{2,1}$, $\hat{Z}_{2,2}$. Assuming $\hat{Z}_{k,n} = 0$ for $n > 4$ and $\hat{Z}_{2,n} = \hat{Z}_{2,4-n}$, as discussed in the previous paragraph, one can compute the full 2 instanton partition function for $SO(7)$ gauge theory at $n_{\mathbf{8}} = 1$,

$$Z_{k=2} = e^{2m}\sum_{n=0}^{4} e^{-nm}\hat{Z}_{k=2,n}\ . \tag{4.2.35}$$

We have checked that this completely agrees with our index of section 2.

Next we consider the instanton quantum mechanics of 5d $SO(7)$ gauge theory with two hypermultiplets. From Fig. 4.8, the contour integrand $Z_{\text{1-loop}}$ of $k$

instantons with $n_1$ and $n_2$ hypermultiplet particles is given by

$$[Z_{\text{1-loop}}]^{SO(7)}_{k,n_1,n_2}(\phi_i,\chi_I,\chi'_{I'}) = \frac{[Z_{\text{1-loop}}]^{SO(7)}_{k,n_1}(\phi_i,\chi_I)\cdot[Z_{\text{1-loop}}]^{SO(7)}_{k,n_2}(\phi_i,\chi'_{I'})}{[Z_{\text{1-loop}}]^{SO(7)}_{k,0}(\phi_i)} \tag{4.2.36}$$

where $[Z_{\text{1-loop}}]^{SO(7)}_{k,n}$ is given by (4.2.23), (4.2.24). Here $i=1,\cdots,k$ is the $Sp(k)$ index, $I=1,\cdots,n_1$ and $I'=1,\cdots,n_2$ are $O(n_1)$ and $O(n_2)$ indices respectively. We summarize the results of our calculations:

$$\begin{aligned}
&Z^{SO(7)}_{1,0,0} = \hat{Z}^{SO(7)}_{1,0,2} = \hat{Z}^{SO(7)}_{1,2,0} = Z^{SO(7)}_{1,0} \qquad (4.2.37)\\
&\hat{Z}^{SO(7)}_{1,1,0} = \hat{Z}^{SO(7)}_{1,0,1} = \hat{Z}^{SO(7)}_{1,1,2} = \hat{Z}^{SO(7)}_{1,2,1} = \hat{Z}^{SO(7)}_{1,1}\\
&\hat{Z}^{SO(7)}_{1,1,1} = \frac{t}{(1-tu^{\pm1})}\prod_{i<j}\frac{t^4}{(1-t^2b_i^{\pm}b_j^{\pm})}\big[\chi^{SU(2)}_{\mathbf{9}} + \chi^{SU(2)}_{\mathbf{7}}(\chi^{SO(7)}_{\mathbf{35}} + \chi^{SO(7)}_{\mathbf{7}} + 1)\\
&+\chi^{SU(2)}_{\mathbf{5}}(-\chi^{SO(7)}_{\mathbf{105}}+1) + \chi^{SU(2)}_{\mathbf{3}}(-\chi^{SO(7)}_{\mathbf{168'}} + \chi^{SO(7)}_{\mathbf{77}} - \chi^{SO(7)}_{\mathbf{21}}) + \chi^{SO(7)}_{\mathbf{330}} + \chi^{SO(7)}_{\mathbf{189}} + \chi^{SO(7)}_{\mathbf{27}}\big]
\end{aligned}$$

where $Z^{SO(7)}_{1,0}$ and $Z^{SO(7)}_{1,1}$ are given by (4.2.26), (4.2.29). With the data shown in (4.2.37), one can compute $Z_{k=1}$ for the $SO(7)$ at $n_{\mathbf{8}}=2$, using the fermion zero mode structures and $Sp(2)$ Weyl symmetry, extending the discussions for $n_{\mathbf{8}}=1$ in the paragraph containing (4.2.33). Namely, at $k$ instanton sector, there are $2k$ fermion zero modes which rotate in $m_1$ and $m_2$, respectively. This means that $(n_1)_{\max}=(n_2)_{\max}=2k$, with zero point energy factor $e^{-\epsilon_0}=e^{k(m_1+m_2)}$ from Weyl symmetry. Weyl symmetry also requires $\hat{Z}_{k,n_1,n_2}=\hat{Z}_{k,2k-n_1,n_2}=\hat{Z}_{k,n_1,2k-n_2}$. (Our calculus on the second line of (4.2.37), relating $\hat{Z}_{1,1,2}$, $\hat{Z}_{1,2,1}$ to other coefficients, partially reconfirms this general argument.) With these structures and (4.2.37), one finds

$$\begin{aligned}
Z_1 &= e^{m_1+m_2}\Big[Z^{SO(7)}_{1,0} + (e^{-m_1}+e^{-m_2})\hat{Z}^{SO(7)}_{1,1} + (e^{-2m_1}+e^{-2m_2})Z^{SO(7)}_{1,0}\\
&\quad + e^{-m_1-m_2}\hat{Z}^{SO(7)}_{1,1,1} + (e^{-2m_1-m_2}+e^{-m_1-2m_2})\hat{Z}^{SO(7)}_{1,1} + e^{-2m_1-2m_2}Z^{SO(7)}_{1,0}\Big]\\
&= \chi^{Sp(2)}_{\mathbf{4}}\hat{Z}^{SO(7)}_{1,1} + \chi^{Sp(2)}_{\mathbf{5}}Z^{SO(7)}_{1,0} + \Big(\hat{Z}^{SO(7)}_{1,1,1} - Z^{SO(7)}_{1,0}\Big)
\end{aligned} \tag{4.2.38}$$

where $\chi_{\mathbf{4}}^{Sp(2)} = \sum_{\pm}(e^{\pm m_1} + e^{\pm m_2})$, $\chi_{\mathbf{5}}^{Sp(2)} = 1 + \sum_{\pm,\pm} e^{\pm m_1 \pm m_2}$. This completely agrees with (4.1.31).

As explained in section 2, one can Higgs the $SO(7)$ gauge theory with a matter hypermultiplet in $\mathbf{8}$, to pure $G_2$ Yang-Mills theory by giving VEV to the hypermultiplet. In the index, this amounts to setting $m_{n_{\mathbf{8}}} = \epsilon_+$, $v_4 = 0$. See section 2.2. Since we have provided concrete tests of $SO(7)$ instanton partition functions of section 2 using our D-brane-based methods, Higgsing both sides do not yield any further significant information or tests. Namely, calculations in this section at $n_{\mathbf{8}} = 1, 2$ already tested our $G_2$ instanton calculus of section 2 at $n_{\mathbf{7}} = 0, 1$. Therefore, we shall not repeat the analysis of Higgsings to $G_2$ in our D-brane-based formalism.

## 4.3 Strings of non-Higgsable 6d SCFTs

In this section, we study the strings of non-Higgsable 6d SCFTs containing $G_2$ theories or $SO(7)$ theories with matters in $\mathbf{8}$. In particular, we shall construct the 2d gauge theories for the strings of 6d atomic SCFTs with 2 and 3 dimensional tensor branches [11].

We first briefly review the 'atomic classification' [6, 7, 11] of 6d $\mathcal{N} = (1, 0)$ SCFTs. This is based on F-theory engineering of 6d SCFTs, on elliptic Calabi-Yau 3-fold ($\text{CY}_3$). Elliptic $\text{CY}_3$ admits a $T^2$ fibration over a 4d base $\mathcal{B}$, which is non-compact and singular. The singular point on $\mathcal{B}$ hosts 6d degrees of freedom which decouple from 10d bulk at low energy. In 6d QFT, resolving this singularity corresponds to going to the tensor branch. Namely, there is a 6d supermultiplet called tensor multiplet, consisting of a self-dual 2-form potential $B_{\mu\nu}$ (whose field strength $H = dB + \cdots$ satisfies $H = \star_6 H$), a real scalar $\Phi$, and fermions. Giving VEV to $\Phi$, one goes into the tensor branch. Geometri-

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 12 |
|---|---|---|---|---|---|---|---|---|---|
| gauge symmetry | - | - | $SU(3)$ | $SO(8)$ | $F_4$ | $E_6$ | $E_7$ | $E_7$ | $E_8$ |
| global symmetry | $E_8$ | $SO(5)_R$ | - | - | - | - | - | - | - |
| matter | | | - | - | - | - | $\frac{1}{2}\mathbf{56}$ | - | - |

Table 4.3: Symmetries/matters of SCFTs with rank 1 tensor branches

| base | $3, 2$ | $3, 2, 2$ | $2, 3, 2$ |
|---|---|---|---|
| gauge symmetry | $G_2 \times SU(2)$ | $G_2 \times SU(2) \times \{\ \}$ | $SU(2) \times SO(7) \times SU(2)$ |
| matter | $\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$ | $\frac{1}{2}(\mathbf{7}+\mathbf{1},\mathbf{2})$ | $\frac{1}{2}(\mathbf{2},\mathbf{8},\mathbf{1})+\frac{1}{2}(\mathbf{1},\mathbf{8},\mathbf{2})$ |

Table 4.4: Non-Higgsable atomic SCFTs with higher rank tensor branches

cally, the singularity of $\mathcal{B}$ is resolved into a collection of intersecting 2-cycles $\mathbb{P}^1$. Associated with the $i$'th $\mathbb{P}^1$, there is a tensor multiplet $B^i$, $\Phi^i$, and sometimes a non-Abelian vector multiplet $A^i$ with simple gauge group $G_i$. The VEV of $\Phi^i$ is proportional to the volume of the $i$'th $\mathbb{P}^1$. Depending on how the 2-cycles intersect, the vector multiplets form a sort of 'quiver' possibly with charged hypermultiplet matters. Geometrically, the vector and hypermultiplets are determined by how the $T^2$ fiber degenerates on $\mathcal{B}$. Equivalently, they depend on the 7-branes wrapping $\mathcal{B}$. With a given resolution of the singularity on $\mathcal{B}$, there are families of theories related to others by Higgsings. The classification of [6, 7, 11] proceeds by first identifying possible non-Higgsable theories, and then considering possible 'un-Higgsings.'

Non-Higgsable theories are constructed by first taking a finite set of 'quiver nodes' and connecting them with certain rules. Technically, the nodes are connected by suitably gauging the E-string theory and identifying them with the gauge groups of the quiver nodes. See [6] for the detailed rules. Roughly speaking, the possible 'quiver nodes' are given in Tables 4.3 and 4.4. More precisely, the SCFTs at $n = 1$ and $n = 2$ play different roles: see [6, 7] for the precise ways of using the SCFTs in Table 4.3, 4.4. The SCFTs in Table 4.3 are called 'minimal SCFTs' in [15]. Here, the numbers on the first rows denote the self-intersection numbers of $\mathbb{P}^1$. Thus in Table 4.4, there are two or three 2-cycles (tensor multiplets).

We are interested in the self-dual strings, which are charged under $B^i_{\mu\nu}$ with equal electric and magnetic charges. If a node has gauge symmetry, the string is identified as an instanton string soliton. See, e.g. [5] and references therein for a review. In this section, we are interested in the strings of the SCFTs given in Table 4.4. Since they involve $G_2$ gauge group with matters in **7** or $SO(7)$ gauge group with matters in **8**, the gauge theories on these strings will be constructed using our gauge theories of section 2 as ingredients.

#### 4.3.1 $2, 3, 2$: $SU(2) \times SO(7) \times SU(2)$ gauge group

Since this QFT has three factors of simple gauge groups, one can assign three topological numbers $k_1, k_2, k_3$ for the instanton strings in $SU(2)_1$, $SO(7)$, $SU(2)_2$. To construct the 2d quiver for these strings, we proceed in steps. We first consider the case in which two of the three gauge symmetries are ungauged in 6d, when only one of $k_1, k_2, k_3$ is nonzero. They are instanton strings of either $SU(2)$ or $SO(7)$ gauge theory with certain matters. After identifying three ADHM(-like) gauge theories, we then consider the case with all $k_1, k_2, k_3$ nonzero, and form a quiver of the three ADHM(-like) theories.

We first consider the case with $k_1 = k_3 = 0$, when $SU(2)_1 \times SU(2)_2$ is ungauged. Then $SU(2)^2 \sim Sp(1)^2$ becomes a flavor symmetry rotating the hypermultiplets, which in the strict ungauging limit enlarges to $Sp(2)$. This is because the matters in $\frac{1}{2}(\mathbf{2}, \mathbf{8}, \mathbf{1}) + \frac{1}{2}(\mathbf{1}, \mathbf{8}, \mathbf{2})$ will arrange into $\frac{1}{2}(\mathbf{8}, \mathbf{4})$ of $SO(7) \times Sp(2)$ in the ungauging limit. This theory was discussed in section 2.1, the 6d $SO(7)$ theory at $n_{\mathbf{8}} = 2$. So as the ADHM-like description, we take this theory with $U(k_2)$ gauge symmetry and reduced $SU(4) \times U(2) \subset SO(7) \times Sp(2)$ global symmetry. Note that in section 2, our 2d gauge theory can have $U(4)$ global symmetry rotating 4 Fermi multiplets, but it reduced to $U(2)$ after coupling to the 5d/6d background fields, especially the hypermultiplet scalar VEV. So the relevant global symmetry of this model (as describing higher dimensional QFT's soliton) depends on the bulk information. Here, since we shall use this model for the strings of the non-Higgsable $2,3,2$ SCFT, with $SU(2)^2$ gauged, one cannot turn on such a background hypermultiplet field. Instead, $SU(2)^2 \subset U(4)$ global symmetry will remain in 2d after 6d gauging. 4 Fermi fields are divided into 2 pairs, and we can rotate them only within a pair.

We also consider the limit in which $SO(7) \times SU(2)_2$ is ungauged, and consider $k_1$ instanton strings in $SU(2)_1$. The matter $\frac{1}{2}(\mathbf{1}, \mathbf{8}, \mathbf{2})$ will not affect the ADHM construction since it is neutral in $SU(2)_1$. $\frac{1}{2}(\mathbf{2}, \mathbf{8}, \mathbf{1})$ will reduce to four fundamental hypermultiplets in $SU(2)$. Its ADHM construction is well known. The 2d $(0,4)$ field contents are given as follows:

$$
\begin{aligned}
(A_\mu, \lambda_0, \lambda) &\quad : \quad \text{vector mutiplet in } (\mathbf{adj}, \mathbf{1}) \\
q_{\dot\alpha} = (q, \tilde{q}^\dagger) &\quad : \quad \text{hypermultiplet in } (\mathbf{k}, \bar{\mathbf{2}}) \\
a_{\alpha\dot\beta} \sim (a, \tilde{a}^\dagger) &\quad : \quad \text{hypermultiplet in } (\mathbf{adj}, \mathbf{1}) \\
\Psi_a &\quad : \quad \text{Fermi multiplet in } (\mathbf{k}, \mathbf{1}) \ ,
\end{aligned}
\tag{4.3.1}
$$

where $a = 1, \cdots, 4$. We showed the representations of $U(k_1) \times SU(2)$. As for the hypermultiplets, we have only shown the scalar components. $\alpha, \dot\alpha = 1, 2$ are the doublet indices for $SU(2)_l$ and $SU(2)_r$. Although $\bar{\mathbf{2}} \sim \mathbf{2}$ for $SU(2)$, we put bar since the ADHM construction classically has $U(2)$ symmetry as a default. This is the UV quiver description for the $SU(2)$ instanton string at $n_{\mathbf{2}} = 4$. This quiver classically has $U(k_1)$ gauge symmetry and $U(2) \times U(4)_F$ global symmetries.

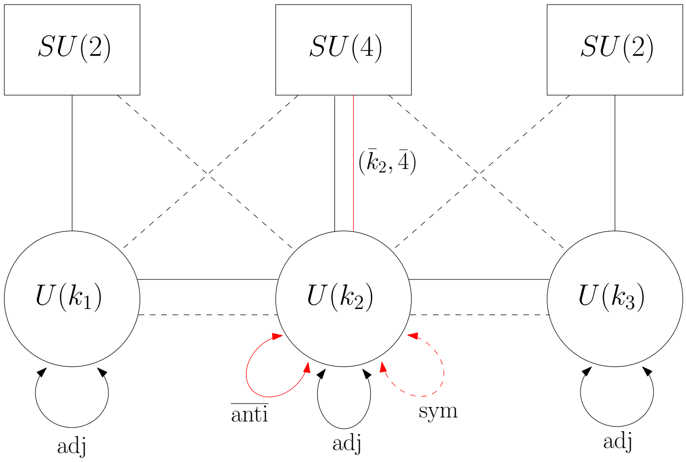



Figure 4.10: 2d quiver for the strings of 6d 2, 3, 2 SCFT. Black lines are fields taking the form of $\mathcal{N} = (0, 4)$ multiplets, being either hypermultiplet/twisted hypermultiplet (bold line) or Fermi multiplet (dashed). Red lines are $\mathcal{N} = (0, 2)$ chiral(bold)/Fermi(dashed) multiplets.

$U(k_1)$ is anomaly-free [13]. The overall $U(1)_G \subset U(2)$ and $U(1)_F \subset U(4)_F$ has mixed anomaly with $U(1) \subset U(k_1)$, and only $G+F$ is free of mixed anomaly [13]. Moreover, considering all fields in this ADHM quiver, $G + F$ can be eaten up by $U(1) \subset U(k_1)$. This implies that $U(k_1)$ gauge invariant observables will not see $G, F$. So this system only has $SU(2) \times SU(4)_F$ symmetry [13]. In the IR, this enhances to $SU(2) \times SO(7)_F$. This is in contrast to the $SU(2)$ theory at $n_{\mathbf{2}} = 4$ in lower dimensions, in which case $U(4)_F$ enhances to $SO(8)$. The $SO(7)_F$ symmetry of this model was noticed in [7,54]. Replacing $k_1$ by $k_3$, one can also obtain the ADHM gauge theory when $SU(2)_1 \times SO(7)$ is ungauged in 6d.

Now when all $k_1, k_2, k_3$ are nonzero, one can form a quiver of the above three ADHM(-like) theories. We shall add more 2d matters to account for the zero modes coming from 6d hypermultiplets, and introduce extra potentials. Between adjacent $SU(2)_1 \times SO(7)$ or $SO(7) \times SU(2)_2$ pair of nodes, one has bi-fundamental hypermultiplet in $\frac{1}{2}(\mathbf{8}, \mathbf{2})$. Since we seek for a 2d UV description seeing $SU(2) \times SU(4)$ subgroup only, this hypermultiplet is in $(\mathbf{2}, \bar{\mathbf{4}})$ bi-fundamental representation of the latter. Usually in D-brane models with bi-fundamental matters, the induced $(0,4)$ matters on the $U(k_1) \times U(k_2)$ ADHM construction of instantons are

$$\begin{aligned} \Phi_A = (\Phi, \tilde{\Phi}^\dagger) \quad &: \quad \text{twisted hypermultiplet in } (\mathbf{k}_1, \overline{\mathbf{k}}_2) \\ \Psi_\alpha = (\Psi_1, \Psi_2) \quad &: \quad \text{two Fermi multiplet fields in } (\mathbf{k}_1, \overline{\mathbf{k}}_2) \end{aligned} \tag{4.3.2}$$

and

$$\begin{aligned} \Psi_a \quad &: \quad \text{Fermi multiplets in } (\mathbf{k}_1, \overline{\mathbf{4}}) \text{ of } U(k_1) \times SU(4) \quad (a = 1, \cdots, 4) \\ \Psi_i \quad &: \quad \text{Fermi multiplets in } (\overline{\mathbf{k}}_2, \mathbf{2}) \text{ of } U(k_2) \times SU(2) \quad (i = 1, 2)\ . \end{aligned} \tag{4.3.3}$$

See, e.g. [13, 16] for the details. Although our construction is not guided by D-brane models, we advocate the same field contents as our natural ansatz. The fields $\Psi_a$ with $a = 1, \cdots, 4$ are not new, but come from the last line of (4.3.1). This is natural because the 6d $SU(4) \subset SO(7)$ gauge symmetry is obtained by gauging the global symmetry in the setting of (4.3.1). Also, $\Psi_i$ with $i = 1, 2$ can also be found in the ADHM-like quiver in section 2. Namely, in section 2, we had four Fermi multiplet fields in $\mathbf{k_2}$ representation of $U(k_2)$, at $n_{\mathbf{8}} = 2$. $\Psi_i$ of (4.3.3) is obtained by taking two of these four. (The other two will be associated with the $SO(7) \times SU(2)_2$ pair.) The bi-fundamental fields in (4.3.2) are new, and link the two ADHM(-like) gauge nodes. Similarly, between the second and third nodes, bi-fundamental fields of the form of (4.3.2), replacing $k_1 \to k_3$, are added. The remaining Fermi fields in the second and third nodes take the form of (4.3.3), with $k_1 \to k_3$. The flavor symmetry of these Fermi multiplets in an ADHM node is locked with the 6d gauge symmetry of the adjacent ADHM node. The resulting quiver is shown schematically in Fig. 4.10.

In the previous paragraph, and in Fig. 4.10, we locked some 6d flavor symmetries of an ADHM theory with 6d gauge symmetries of adjacent ADHM theories. This has to be justified by writing down the interactions which lock the symmetries as claimed. Now we explain such superpotentials. In the $(0,2)$ off-shell description [46] of $(0,4)$ theories, one can introduce interactions by two kinds of superpotentials $J_\Psi$, $E_\Psi$ given for each Fermi multiplet. There are some constraints on $J_\Psi$'s and $E_\Psi$'s to be met, either for $(0,2)$ SUSY or for $(0,4)$ enhancement of the classical action. These conditions are all mentioned in section 3, when we discussed models with manifest $(0,4)$ SUSY. In our current ADHM-like models, some part of the matters and interactions inevitably break manifest $(0,4)$ SUSY. However, most of the fields still take the form of $(0,4)$ multiplets, so that we find it convenient to turn on classical interactions in two steps. We first turn on manifestly $(0,4)$ supersymmetric classical interactions for the fields shown in Fig. 4.10 with black lines/nodes. Then we rephrase these interactions in $\mathcal{N}=(0,1)$ language, after which we turn on further $(0,1)$ interactions for the fields shown as red lines in Fig. 4.10. We find that securing the partial $(0,4)$ SUSY structure plays important roles for the correct physics, e.g. yielding the right multi-particle structures of the elliptic genus, etc.

In $(0,4)$ gauge theories, one has two types of hypermultiplets: hypermultiplet whose scalars form a doublet of $SU(2)_r$, and twisted hypermultiplet whose scalars form a doublet of $SU(2)_R$. These two multiplets contribute differently to the $J,E$ superpotentials for the fermions in the $(0,4)$ vector multiplet. Namely, in the $(0,2)$ formalism of [46], a $(0,4)$ vector multiplet decomposes into a $(0,2)$ vector multiplet $A_\mu,\lambda_0$ and an adjoint Fermi multiplet $\lambda$ (plus auxiliary field). A hypermultiplet field $(\Phi_{\dot\alpha})_{\mathbf{R}}=(\Phi,\tilde\Phi^\dagger)_{\mathbf{R}}$ in the representation $\mathbf{R}$ of the gauge group contributes $J_{\lambda^a}=\Phi_{\mathbf{R}}[T^a_{\mathbf{R}}]\tilde\Phi_{\mathbf{R}}$. A twisted hypermultiplet $(\Phi_A)_{\mathbf{R}}=(\Phi,\tilde\Phi^\dagger)_{\mathbf{R}}$ contributes to $E_{\lambda^a}=\Phi_{\mathbf{R}}[T^a_{\mathbf{R}}]\tilde\Phi_{\mathbf{R}}$. This is the requirement of $(0,4)$ supersymmetry. (In our normalization of section 3, one has $\sqrt{2}$ factors multiplied.) However, from the $(0,2)$ SUSY, they should satisfy $\sum_\Psi J_\Psi E_\Psi = 0$. To meet this condition, one has to turn on extra potentials for the Fermi multiplets shown as black lines in Fig. 4.10. This is in complete parallel with the results shown in section 3. Let us name the fields in Fig. 4.10 with black lines/nodes as follows. The ADHM fields within an ADHM node are named as

$$\begin{aligned}
\text{node 1} \quad &: \quad q_1, \tilde{q}_1 \in (\mathbf{k}_1, \bar{\mathbf{2}}_1) + (\bar{\mathbf{k}}_1, \mathbf{2}_1) \ , \quad a, \tilde{a} \in \mathbf{adj}_1 \\
\text{node 2} \quad &: \quad q_2, \tilde{q}_2 \in (\mathbf{k}_2, \bar{\mathbf{4}}) + (\bar{\mathbf{k}}_2, \mathbf{4}) \ , \quad a, \tilde{a} \in \mathbf{adj}_2 \\
\text{node 3} \quad &: \quad q_3, \tilde{q}_3 \in (\mathbf{k}_3, \bar{\mathbf{2}}_3) + (\bar{\mathbf{k}}_3, \mathbf{2}_3) \ , \quad a, \tilde{a} \in \mathbf{adj}_3 \ ,
\end{aligned} \tag{4.3.4}$$

while the fields linking the adjacent nodes are named as

$$\begin{aligned}
\text{link 1-2} \quad &: \quad \Phi_{12}, \tilde{\Phi}_{12} \in (\mathbf{k}_1, \bar{\mathbf{k}}_2) + (\bar{\mathbf{k}}_1, \mathbf{k}_2) \ , \quad \Psi_{12}, \tilde{\Psi}_{12} \in (\mathbf{k}_1, \bar{\mathbf{k}}_2) + (\bar{\mathbf{k}}_1, \mathbf{k}_2) \\
&\quad\ \ \psi_{12}, \tilde{\psi}_{12} \in (\mathbf{k}_1, \bar{\mathbf{4}}) + (\bar{\mathbf{k}}_2, \mathbf{2}_1) \\
\text{link 2-3} \quad &: \quad \Phi_{23}, \tilde{\Phi}_{23} \in (\mathbf{k}_2, \bar{\mathbf{k}}_3) + (\bar{\mathbf{k}}_2, \mathbf{k}_3) \ , \quad \Psi_{23}, \tilde{\Psi}_{23} \in (\mathbf{k}_2, \bar{\mathbf{k}}_3) + (\bar{\mathbf{k}}_2, \mathbf{k}_3) \\
&\quad\ \ \psi_{23}, \tilde{\psi}_{23} \in (\mathbf{k}_2, \bar{\mathbf{2}}_3) + (\bar{\mathbf{k}}_3, \mathbf{4}) \ .
\end{aligned} \tag{4.3.5}$$

Here, notations like $\mathbf{2}_1$, $\mathbf{2}_3$ mean representations of $SU(2)$ on the first (leftmost) and the third (rightmost) nodes, respectively. Then, using the results of [55], eqns. (3.3) and (3.4), we find the following superpotentials after mapping our fields with those in Table 4 of [55]:

$$\begin{aligned}
\text{nodes} \quad &: \quad J_{\lambda_i} = \sqrt{2}(q_i\tilde{q}_i + [a_i, \tilde{a}_i]) \quad (\text{for } i = 1,2,3) \ , \quad E_{\lambda_1} = \sqrt{2}\Phi_{12}\tilde{\Phi}_{12} \ , \\
& \quad E_{\lambda_2} = \sqrt{2}(\Phi_{23}\tilde{\Phi}_{23} - \tilde{\Phi}_{12}\Phi_{12}) \ , \quad E_{\lambda_3} = -\sqrt{2}\tilde{\Phi}_{23}\Phi_{23} \\
\text{links} \quad &: \quad E_{\Psi_{i-1,i}} = \sqrt{2}(\Phi_{i-1,i}a_i - a_{i-1}\Phi_{i-1,i}) \ , \quad J_{\Psi_{i-1,i}} = \sqrt{2}(\tilde{a}_i\tilde{\Phi}_{i-1,i} - \tilde{\Phi}_{i-1,i}\tilde{a}_{i-1}) \ , \\
& \quad E_{\tilde{\Psi}_{i-1,i}} = \sqrt{2}(\tilde{a}_{i-1}\Phi_{i-1,i} - \Phi_{i-1,i}\tilde{a}_i) \ , \quad J_{\tilde{\Psi}_{i-1,i}} = \sqrt{2}(a_i\tilde{\Phi}_{i-1,i} - \tilde{\Phi}_{i-1,i}a_{i-1}) \ , \\
& \quad E_{\psi_{i-1,1}} = \sqrt{2}\Phi_{i-1,i}q_i \ , \quad J_{\psi_{i-1,i}} = \sqrt{2}\tilde{q}_i\tilde{\Phi}_{i-1,i} \\
& \quad E_{\tilde{\psi}_{i-1,1}} = \sqrt{2}\tilde{q}_{i-1}\Phi_{i-1,i} \ , \quad J_{\tilde{\psi}_{i-1,i}} = -\sqrt{2}\tilde{\Phi}_{i-1,i}q_{i-1} \quad (\text{for } i = 2,3) \ . \qquad (4.3.6)
\end{aligned}$$

(We correct overall normalization of [55] by $\sqrt{2}$ factors.) These are part of the interactions, and we shall add more interactions later preserving less SUSY. Only with the interactions shown above, one can check the $(0,4)$ SUSY of the classical action, for instance in the bosonic potential [46,55]. The rearrangement of the potential energy with $SU(2)_r \times SU(2)_R$ symmetry can be made similar to eqn.(3.6) of [55]. In particular, the flavor symmetries which rotate Fermi multiplets are locked by these interactions as shown in Fig. 4.10.

We now proceed to write down all the interactions preserving only $(0,1)$ symmetry, for the red fields associated with the middle '3' node. This will basically be the same as the interactions explained in section 2.1, for $SO(7)$ instanton strings at $n_{\mathbf{8}} \neq 0$. However, before doing that, we should rephrase the previous $(0,4)$ interactions in the $(0,1)$ superfield language. In $(0,2)$ superfield, one has a pair of complex superspace coordinates $\theta, \bar{\theta}$. $E_\Psi$ appears as the top component $\sim \theta\bar{\theta}E_\Psi(\Phi)$ of the Fermi multiplet [47]. On the other hand, $J_\Psi$ appears as a term in the Lagrangian, of the form $\int d\theta \Psi J_\Psi + h.c.$. However, since $(0,1)$ supersymmetry only has one real superspace coordinate $\theta$, there is no separate notion of $E_\Psi$. There can be superpotentials $\int d\theta(\Psi J_\Psi^{(0,1)} - h.c.)$, where $J_\Psi^{(0,1)}$ can be any non-holomorphic function of the scalars. To realize $J_\Psi$

and $E_\Psi$ in the previous paragraph, one writes

$$\sum_\Psi \int d\theta \left[\Psi(J_\Psi(\Phi) + \bar{E}_\Psi(\bar{\Phi})) - \text{h.c.}\right] \ . \tag{4.3.7}$$

One finds the correct bosonic potential $\sum_\Psi \left|J_\Psi + \bar{E}_\Psi\right|^2 = \sum_\Psi \left(|J_\Psi|^2 + |E_\Psi|^2\right)$, using $\sum_\Psi J_\Psi E_\Psi = 0$ of (4.3.6). The Yukawa couplings associated with $J_\Psi$ and $E_\Psi \sim \sum_\Psi \Psi(\frac{\partial J_\Psi}{\partial \phi^i}\psi^i + \frac{\partial \bar{E}}{\partial \bar{\phi}_i}\psi_i)$ are also correctly reproduced. Now with (4.3.6) rewritten as $J^{(0,1)}_\Psi = J_\Psi + \bar{E}_\Psi$, we add further interactions for $\hat{\lambda}$, $\check{\lambda}$ on the middle node, as given by (4.1.7).

With these potentials, one can show that the moduli space is that of each ADHM-like quiver, at $\Phi_{i-1,i} = 0$, $\tilde{\Phi}_{i-1,i} = 0$. In particular, no extra branch is formed by $\Phi_{i-1,i}, \tilde{\Phi}_{i-1,i}$.

One can compute the 2d anomalies from our gauge theory, and compare with the result known from anomaly inflow. The 6d 1-loop anomaly 8-form in the tensor branch is given by

$$\begin{aligned} I_{\text{1-loop}} &= -\frac{3}{32}\left[\text{Tr}(F^2_{SO(7)})\right]^2 - \frac{1}{16}\left[\text{Tr}(F^2_{SU(2)_1})\right]^2 - \frac{1}{16}\left[\text{Tr}(F^2_{SU(2)_2})\right]^2 \\ &\quad + \frac{1}{16}\text{Tr}(F^2_{SO(7)})\left[\text{Tr}(F^2_{SU(2)_1}) + \text{Tr}(F^2_{SU(2)_2})\right] - \frac{1}{16}p_1(T)\text{Tr}(F^2_{SO(7)}) \\ &\quad - \frac{1}{4}c_2(R)\left[5\text{Tr}(F^2_{SO(7)}) + 2\text{Tr}(F^2_{SU(2)_1}) + 2\text{Tr}(F^2_{SU(2)_2})\right] + \cdots \ . \end{aligned} \tag{4.3.8}$$

We only showed the terms containing $SU(2)_1 \times SO(7) \times SU(2)_2$ gauge fields. This can be written as $I_{\text{1-loop}} = -\frac{1}{2}\Omega^{ij} I_i I_j + \cdots$, with $i, j = 1, 2, 3$, where

$$\Omega^{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \ , \quad I_i = \begin{pmatrix} \frac{1}{4}\text{Tr}(F^2_{SU(2)_1}) + \frac{11}{4}c_2(R) + \frac{1}{16}p_1(T) \\ \frac{1}{4}\text{Tr}(F^2_{SO(7)}) + \frac{7}{2}c_2(R) + \frac{1}{8}p_1(T) \\ \frac{1}{4}\text{Tr}(F^2_{SU(2)_2}) + \frac{11}{4}c_2(R) + \frac{1}{16}p_1(T) \end{pmatrix} . \tag{4.3.9}$$

Using (4.1.40), one finds the following anomaly 4-form $I_4$

$$I_4 = \left(k_1 k_2 + k_2 k_3 - k_1^2 - \frac{3}{2}k_2^2 - k_3^2\right)\chi(T_4) + k_2(I_1 + I_3 - 3I_2) + k_1(I_2 - 2I_1) + k_3(I_2 - 2I_3) \, . \tag{4.3.10}$$

on the instanton strings with string numbers $k_i = (k_1, k_2, k_3)$.

We now compute the anomaly from our gauge theory. We first compute the anomalies of three ADHM quivers $I_4^{(i)}$ $(i = 1, 2, 3)$, restricting them according to the symmetry locking rules. We then compute the anomalies $I_4^{\text{bif}}$ of matters $\Phi_{i-1,i}, \tilde{\Phi}_{i-1,i}$. The net anomaly is $I_4 = \sum_{i=1}^{3} I_4^{(i)} + I_4^{\text{bif}}$. Using (4.1.41), one first finds

$$I_4^{(2)} = -\frac{3}{2}k_2^2\chi(T_4) - k_2\left[\frac{3}{4}\text{Tr}(F_{SO(7)}^2) + 5c_2(R) + \frac{p_1(T)}{4} - \frac{1}{4}(\text{Tr}(F_{SU(2)_1}^2) + \text{Tr}(F_{SU(2)_2}^2))\right] \tag{4.3.11}$$

where we replaced $\text{tr}_{\mathbf{4}}(F_{Sp(2)}^2) \to \text{tr}_{\mathbf{2}}(F_{SU(2)_1}^2) + \text{tr}_{\mathbf{2}}(F_{SU(2)_2}^2) = \frac{1}{2}[\text{Tr}(F_{SU(2)_1}^2) + \text{Tr}(F_{SU(2)_2}^2)]$. As in section 2.1, $F_{SO(7)}$ is restricted to $SU(4)$ in our UV gauge theory, and fields in $c_2(R)$, $c_2(r)$ are also restricted to $F_J$. $I_4^{(1)}$ and $I_4^{(3)}$ can be computed from the known anomaly polynomial for the instanton strings of 6d $SU(2)$ theory at $n_{\mathbf{2}} = 4$. The result is eqn.(5.19) of [5] at $N = 2$, with $k$ replaced by $k_1$ or $k_3$:

$$I_4^{(1)} = -k_1^2\chi(T_4) - \frac{k_1}{2}\text{Tr}(F_{SU(2)_1}^2) + \frac{k_1}{4}\text{Tr}(F_{SO(7)}^2) - 2k_1c_2(R)\ , \quad I_4^{(3)} = (k_1, SU(2)_1 \to k_3, SU(2)_2) \tag{4.3.12}$$

Here we replaced $F = SU(4)$ of [5] by $SO(7)$, assuming symmetry enhancement. Finally, $I_4^{\text{bif}}$ is also computed in [5], eqn.(3.58), which for our model is

$$I_4^{\text{bif}} = (k_1k_2 + k_2k_3)\chi(T_4)\ . \tag{4.3.13}$$

One finds that $I_4 = \sum_{i=1}^{3} I_4^{(i)} + I_4^{\text{bif}}$ agrees with (4.3.10), providing a check of our gauge theory.

The elliptic genus of this gauge theory is given by (note again the definition

$\theta(z) \equiv \frac{i\theta_1(\tau|\frac{z}{2\pi i})}{\eta(\tau)})$

$$\begin{aligned} Z_{k_1,k_2,k_3} &= \oint \prod_{I_1=1}^{k_1} \frac{\prod_{i=1}^4 \theta(v_i - u_{I_1})}{\theta(\epsilon_+ \pm u_{I_1} \pm \nu)} \cdot \frac{\prod_{I_1\neq J_1} \theta(u_{I_1J_1}) \prod_{I_1,J_1} \theta(2\epsilon_+ + u_{I_1J_1})}{\prod_{I_1,J_1=1}^{k_1} \theta(\epsilon_{1,2} + u_{I_1J_1})} \cdot \Big(1 \to 3, \nu \to \tilde{\nu}\Big) \\ &\cdot \prod_{I_2=1}^{k_2} \frac{\theta(v \pm u_{I_2})\theta(\tilde{v} \pm u_{I_2})}{\prod_{i=1}^4 \theta(\epsilon_+ \pm (u_{I_2} - v_i))\theta(\epsilon_+ - u_{I_2} - v_i)} \cdot \frac{\prod_{I_2\neq J_2} \theta(u_{I_2J_2}) \prod_{I_2,J_2} \theta(2\epsilon_+ + u_{I_2J_2})}{\prod_{I_2,J_2=1}^{k_2} \theta(\epsilon_{1,2} + u_{I_2J_2})} \\ &\cdot \frac{\prod_{I_2\leq J_2} \theta(u_{I_2}+u_{J_2})\theta(u_{I_2}+u_{J_2}-2\epsilon_+)}{\prod_{I_2<J_2} \theta(\epsilon_{1,2}-u_{I_2}-u_{J_2})} \\ &\cdot \prod_{I_1=1}^{k_1} \prod_{I_2=1}^{k_2} \frac{\theta(\epsilon_- \pm (u_{I_1} - u_{I_2}))}{\theta(-\epsilon_+ \pm (u_{I_1} - u_{I_2}))} \cdot \prod_{I_2=1}^{k_2} \prod_{I_3=1}^{k_3} \frac{\theta(\epsilon_- \pm (u_{I_2} - u_{I_3}))}{\theta(-\epsilon_+ \pm (u_{I_2} - u_{I_3}))} \ . \end{aligned} \quad (4.3.14)$$

$v_i$ (with $\sum_i v_i = 0$) is the $SU(4) \subset SO(7)$ chemical potential, $\pm\nu$ and $\pm\tilde{\nu}$ are the chemical potentials for 6d $SU(2)^2$. The contour integral is given with suitable weight [29], including the $U(k_1) \times U(k_2) \times U(k_3)$ Weyl factor. The contour integral is again given by the JK residues [29]. We again choose $\eta_1 = (1, \cdots, 1)$, $\eta_2 = (1, \cdots, 1)$, $\eta_3 = (1, \cdots, 1)$. Then, similar to the residue choices made in section 2, one can show that the residues are labeled by three sets of colored Young diagrams, $(Y_1^{(1)}, Y_2^{(1)})$ with $k_1$ boxes for $u_{I_1}$, $(Y_1^{(2)}, \cdots, Y_4^{(2)})$ with $k_2$ boxes for $u_{I_2}$, and $(Y_1^{(3)}, Y_2^{(3)})$ with $k_3$ boxes for $u_{I_3}$. The residues all come from the poles at

$$\begin{aligned} u_{I_1} &: \ \epsilon_+ + u_{I_1} \pm \nu = 0 \ , \quad \epsilon_{1,2} + u_{I_1J_1} = 0 \\ u_{I_2} &: \ \epsilon_+ + u_{I_2} - v_i = 0 \ , \quad \epsilon_{1,2} + u_{I_2J_2} = 0 \\ u_{I_3} &: \ \epsilon_+ + u_{I_3} \pm \tilde{\nu} = 0 \ , \quad \epsilon_{1,2} + u_{I_3J_3} = 0 \ , \end{aligned} \quad (4.3.15)$$

coming from the first, second and third line of (4.3.14), respectively. The residue sum is given by

$$Z_{k_1,k_2,k_3} = \sum_{\substack{Y_{(1,2,3)} \\ |Y_{(a)}|=k_a}} \prod_{i=1}^{2} \Bigg( \prod_{s_1 \in Y_{(1)i}} \frac{\prod_{l=1}^{4} \theta(v_l - \phi(s_1))}{\prod_{j=1}^{2} \theta(E_{ij}(s_1))\theta(E_{ij}(s_1) - 2\epsilon_+)} \Bigg) \times \prod_{i=1}^{2} \Bigg( 1 \to 3 \Bigg) \tag{4.3.16}$$

$$\times \prod_{i=1}^{4} \Bigg( \prod_{s_2 \in Y_{(2)i}} \frac{\theta(2\phi(s_2))\theta(2\phi(s_2) - 2\epsilon_+) \cdot \theta(v \pm \phi(s_2))\theta(\tilde{v} \pm \phi(s_2))}{\prod_{j=1}^{4} \theta(E_{ij}(s_2))\theta(E_{ij}(s_2) - 2\epsilon_+)\theta(\epsilon_+ - \phi(s_2) - v_j)} \times$$

$$\prod_{s_2 \in Y_{(2)i}} \prod_{j \geq i}^{4} \prod_{\substack{\tilde{s}_2 \in Y_{(2)j} \\ s_2 < \tilde{s}_2}} \frac{\theta(\phi(s_2) + \phi(\tilde{s}_2))\theta(\phi(s_2) + \phi(\tilde{s}_2) - 2\epsilon_+)}{\theta(\epsilon_{1,2} - \phi(s_2) - \phi(\tilde{s}_2))} \times$$

$$\prod_{j=1}^{2} \prod_{s_1 \in Y_{(1)j}} \frac{\theta(-\epsilon_- \pm (\phi(s_1) - \phi(s_2)))}{\theta(-\epsilon_+ \pm (\phi(s_1) - \phi(s_2)))} \times \prod_{j=1}^{2} \prod_{s_3 \in Y_{(3)j}} \frac{\theta(-\epsilon_- \pm (\phi(s_2) - \phi(s_3)))}{\theta(-\epsilon_+ \pm (\phi(s_2) - \phi(s_3)))}$$

where $s_a$ (for $a = 1, 2, 3$) labels the $k_a$ boxes in the $a$'th colored Young diagram, and $(v_{(1)})_{1,2} = \pm\nu$, $(v_{(3)})_{1,2} = \pm\tilde{\nu}$. $\phi(s_a)$ and $E_{ij}(s_a)$ are defined as

$$E_{ij}(s_a) = v_{(a)i} - v_{(a)j} - \epsilon_1 h_i(s_a) + \epsilon_2(v_j(s_a) + 1) \tag{4.3.17}$$

$$\phi(s_a) = v_{(a)i} - \epsilon_+ - (n_a - 1)\epsilon_1 - (m_a - 1)\epsilon_2 \tag{4.3.18}$$

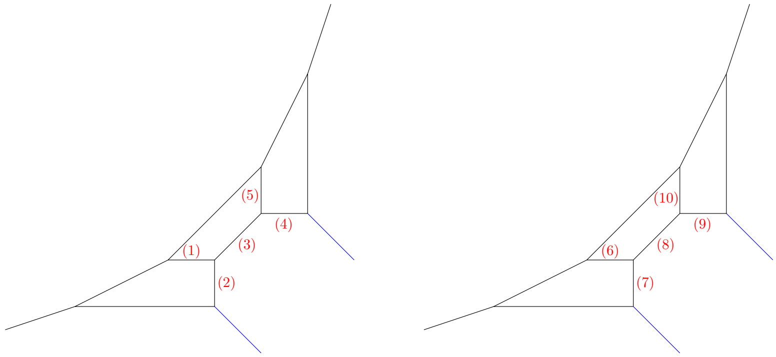
for $s_a = (m_a, n_a) \in Y_i^{(a)}$.

It is important to note that $\Phi_{i-1,i}, \tilde{\Phi}_{i-1,i}$ do not provide extra JK-Res, for the following reason. For instance, suppose that we take the 'pole' from $\theta(-\epsilon_+ + u_{I_1} - u_{I_2})^{-1}$ on the fourth line, at $-\epsilon_+ + u_{I_1} - u_{I_2} = 0$ to determine $u_{I_1}$, with $u_{I_2}$ determined from (4.3.15). Suppose that $u_{I_2}$ is determined by $\epsilon_+ + u_{I_2} - v_i = 0$. Then on the first line of (4.3.14), a Fermi multiplet contribution $\theta(v_i - u_{I_1})$ vanishes at the pole, because $u_{I_1} - v_i = (-\epsilon_+ + u_{I_1} - u_{I_2}) + (\epsilon_+ + u_{I_2} - v_i) = 0$. On the other hand, suppose that $u_{I_2}$ is determined by one of $\epsilon_{1,2} + u_{I_2} - u_{J_2} = 0$, with $u_{J_2}$ determined by other equations. Then, from $\Psi_{12}, \tilde{\Psi}_{12}$'s contributions $\theta(\epsilon_- \pm (u_{I_1} - u_{J_2}))$ on the fourth line, one again finds that one of the two $\theta$ factors vanishes at the pole location. Therefore, one finds that the residue vanishes due to the vanishing determinant from certain Fermi multiplet. This idea turns out to hold most generally, so that one can show that the fourth line of (4.3.14) never provides a pole with nonzero JK residue. Based on these observations, one can make a recursive proof of this statement, similar to that made for the 5d $\mathcal{N} = 1^*$ instanton partition function in [22]. Note that the symmetry locking provided by the $(0, 4)$ potentials (4.3.6) played crucial roles for the vanishing of these residues.

### 4.3.2 Tests from 5d descriptions

In this subsection, we test the elliptic genera of section 4.1, using a recently proposed 5d description for the 6d $2, 3, 2$ SCFT compactified on $S^1$ [17]. The description is available when the elliptic $CY_3$ in F-theory admits an orbifold description, of the form $[\mathcal{B} \times T^2]/\Gamma$ with a discrete group $\Gamma$. One can dualize F-theory to M-theory on same $CY_3$. The small $S^1$ limit (together with suitably scaling other massive parameters) on the F-theory side corresponds to the large $T^2$ limit on the M-theory dual. There may be fixed points of $\Gamma$ on $T^2$, as it decompactifies into $\mathbb{R}^2$. Near each fixed point, there exists an interacting 5d SCFTs. So in this 5d limit, one obtains factors of decoupled 5d SCFTs. The 6d KK momentum degrees of freedom can be restored by locking certain global symmetries of these 5d SCFTs and gauging it, so that the instanton quantum number of this 5d gauge theory provides the 6d KK momentum. See [17, 56–58] for the details.

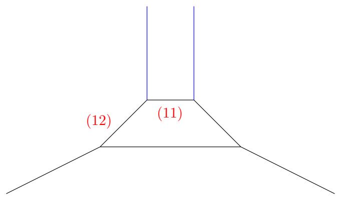
If a 6d SCFT admits a 6d gauge theory description, an obvious 5d limit is given by the 5d gauge theory with same gauge group. This is obtained by a scaling limit with 6d tensor multiplet scalar VEV, $v = \langle \Phi \rangle \to \infty$. Namely, $v = g_{6\mathrm{d}}^{-2}$ and $Rv = g_{5\mathrm{d}}^{-2}$ are the 6d and 5d inverse gauge couplings, respectively,




$$(1):\ \frac{q}{y_1y_2y_3^2},\quad (2):\ e^{-v},\quad (3):\ e^{-u}\,\frac{y_1y_2y_3^2}{\sqrt{q}},\quad (4):\ e^{-\tilde{v}},\quad (5):\ \frac{q}{y_1y_2y_3^2},$$

$$(6):\ \frac{y_2}{y_1},\quad (7):\ \frac{w^2e^{-v}}{y_2y_3},\quad (8):\ e^{-u}\,\frac{y_1y_2^2y_3^3}{w\tilde{w}\sqrt{q}},\quad (9):\ \frac{\tilde{w}^2e^{-\tilde{v}}}{y_2y_3},\quad (10):\ \frac{y_2}{y_1}$$

(a)




$$(11):\ e^{-2u-v-\tilde{v}}\,\frac{y_1^2y_2^2y_3^4}{q},\quad (12):\ y_1y_2$$

(b)

Figure 4.11: 5-brane webs for the 6d 2, 3, 2 SCFT in a 5d limit. (1)∼(12) are the Kahler parameters in terms of our fugacities. $v, u, \tilde{v}$ are tensor VEV's for $SU(2)_1 \times SO(7) \times SU(2)_2$.

where $R$ is the circle radius. If one takes $R \to 0$, $v \to \infty$ with $g_{5\mathrm{d}}^{-2}$ kept fixed, one often gets a 5d SCFT with a relevant deformation made by $g_{5\mathrm{d}}^{-2} \neq 0$ [58].

The 5d factorization limit described in the previous paragraph takes different scaling limit of massive parameters when taking $R \to 0$. The latter 5d limit scales other massive parameters like the holonomies of gauge fields on $S^1$. From the viewpoint of former 5d limit, the latter 5d limit keeps a different slice of 5d states, which contains states with nonzero KK momenta from the former viewpoint. For our $2, 3, 2$ SCFT, the new 5d limit consists of three 5d SCFTs. The three 5d SCFTs admit IIB 5-brane web engineerings, given by Fig. 4.11 [17]. Each factor in Fig. 4.11(a) is a non-Lagrangian theory, in that it does not admit a relevant deformations to 5d Yang-Mills theory. (Fig. 4.11(a) is related to that in [17] by a flop transition.) To have states with general KK momenta, one locks the three $SU(2)_g$ flavor symmetries associated with gauge symmetries on the blue-colored parallel 5-branes of Fig. 4.11, and gauge it. The relations between 6d parameters and the Kahler parameters of 5-brane web are shown below Fig. 4.11, which will be (empirically) justified. [17] also discusses the gauging of $SU(2)_g$ in the brane web context as trivalent gluing, with some prescriptions for computations. But here we shall only discuss computations in the factorization limit.

We want to test our elliptic genera (4.3.14), (4.3.16) using this 5d description. The test will be made in the 5d factorization limit in which $SU(2)_g$ is ungauged, as in Fig. 4.11 with semi-infinite blue lines. In some sectors with special values of $k_1, k_2, k_3$, the BPS spectrum of the brane configuration is well known, so our elliptic genera in these sectors will be tested against known results. More generically, we shall do topological vertex calculus. Technically, identifying the parameters of 6d gauge theory (and our elliptic genus) and those in the 5-brane web is not straightforward. The relation between the two sets of parameters are often determined empirically in the literature. We follow the strategy of [17] which studied the 5d description of 6d gauge theories. [17] used

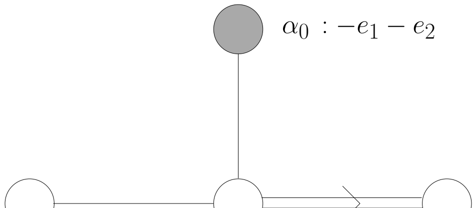


Figure 4.12: Affine Dynkin diagram of $SO(7)$

the guidance from 6d affine gauge symmetry structure to partly determine the relations between 5d/6d parameters, and then empirically fixed the rest. In our problem, we shall use the affine $SO(7)$ symmetry to partly determine the relation, and then focus on well-known subsectors to fix the rest.

We first determine the parameter relation that can be inferred from $SO(7)$ group theory. To this end, we focus on the part of web diagrams of Fig. 4.11 associated with the Kahler parameters (1)=(5), (6)=(10), (12), and the blue 5-branes. Considering how the associated four faces are connected to others (after $SU(2)_g$ gauging), it is natural to conceive that the four Kahler parameters are fugacities for the affine $SO(7)$ symmetry. This is somewhat similar to the identifications of 6d $SU(3)$, $SO(8)$, $E_{6,7,8}$ fugacities in [17], using their affine Dynkin diagrams. For $SO(7)$, the affine Dynkin diagram is given by Fig. 4.12, where $e_1, e_2, e_3$ are orthonormal vectors. We call the fugacities corresponding to the simple roots as $(t_1, t_2, t_3, t_4) \leftrightarrow (\alpha_1, \alpha_0, \alpha_3, \alpha_2)$. From the expressions of the roots in Fig. 4.12, one obtains

$$t_1 = \frac{z_1^2}{z_2^2} \ , \quad t_2 = \frac{q}{z_1^2 z_2^2} \ , \quad t_3 = z_3^2 \ , \quad t_4 = \frac{z_2^2}{z_3^2} \tag{4.3.19}$$

where we used the fact that the KK momentum fugacity $q \equiv e^{2\pi i \tau}$ is associated with $\alpha_0$ in the affine Lie algebra. The root relation $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_0 = 0$ is reflected in the above parameterization as $t_1 t_2 t_3^2 t_4^2 = q$. $z_1, z_2, z_3$ are the fugacities of $SO(7)$ rotating three orthogonal 2-planes. More precisely, the characters of **7** and **8** are given in these parameters by

$$\begin{aligned}\chi_{\mathbf{8}} &= z_1 z_2 z_3 + \frac{z_1 z_2}{z_3} + \frac{z_2 z_3}{z_1} + \frac{z_3 z_1}{z_2} + (\text{inverse of all four terms}) \\ \chi_{\mathbf{7}} &= 1 + z_1^2 + z_2^2 + z_3^2 + (\text{inverse of all three terms})\ .\end{aligned} \tag{4.3.20}$$

The $SU(4)$ fugacity basis $y_i = e^{-v_i}$ $(i = 1, 2, 3)$ that we have been using is related to $z_{1,2,3}$ by $z_1^2 = y_2 y_3$, $z_2^2 = y_3 y_1$, $z_3^2 = y_1 y_2$, so that the characters are given by

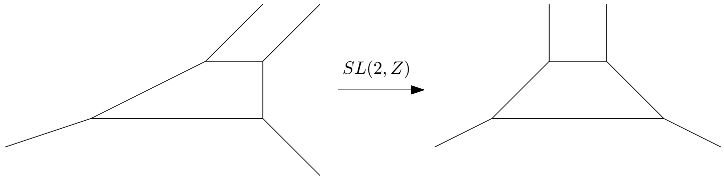
$$\begin{aligned}\chi_{\mathbf{8}} &= y_1 + y_2 + y_3 + \frac{1}{y_1 y_2 y_3} + (\text{inverse of all four terms}) \\ \chi_{\mathbf{7}} &= 1 + y_1 y_2 + y_2 y_3 + y_3 y_1 + (\text{inverse of all three terms})\ .\end{aligned} \tag{4.3.21}$$

$t_{1,2,3,4}$ are given in terms of $y_{1,2,3}$, $q$ by

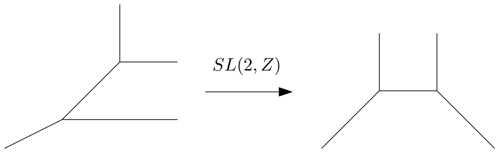
$$t_1 = \frac{y_2}{y_1} = (6) = (10)\ ,\quad t_2 = \frac{q}{y_1 y_2 y_3^2} = (1) = (5)\ ,\quad t_3 = y_1 y_2 = (12)\ ,\quad t_4 = \frac{y_3}{y_2}\ . \tag{4.3.22}$$

$t_4$ is roughly the Kahler parameter for the blue line in Fig. 4.11, which is sent to zero in the factorization limit, with $t_{1,2,3}$ fixed. This limit requires $q \sim y_3^2 \to 0$ with fixed $y_1, y_2$. To fully specify this 5d limit, we still have to specify the scaling of other parameters in $q \to 0$. The remaining parameters are: two $SU(2)$ inverse gauge couplings (or tensor VEV's) which we call $e^{-v}$, $e^{-\tilde{v}}$ in this subsection, two $SU(2)$ fugacities $w, \tilde{w}$ (related to $\nu, \tilde{\nu}$ of section 4.1 by $w = e^{-\nu}$, $\tilde{w} = e^{-\tilde{\nu}}$), $SO(7)$ inverse gauge coupling $e^{-u}$. All the scaling rules except that of $e^{-u}$ will be determined below by considering an $SU(2)$ subsector. The scaling of $e^{-u}$ will then be determined next by considering the $SO(7)$ subsector, at which stage we shall already make some tests of our elliptic genera. Then we consider more general sectors for further tests.

**$SU(2)$ subsector:** We first study the limit in which $SO(7)$ is ungauged, or equivalently, when $k_2 = 0$. The limit $u \to \infty$ should yield two 6d $SU(2)$ the-




(a)




(b)

Figure 4.13: Ingredients of the 5d description of 6d $SU(2)$ theory at $n_{\mathbf{2}} = 4$

ories at $n_{\mathbf{2}} = 4$, decoupled to each other. So in this limit, the brane web of Fig. 4.11 (with $SU(2)_g$ gauged) should factorize into two. The natural identification of $u \to \infty$ in the web is to take the distance between the parallel blue lines to infinity. (Assuming the identification of Kahler parameters in Fig. 4.11, the distance between two blue lines is proportional to $(11) = (2)(3)^2(4) = (7)(8)^2(9) \propto e^{-2u}$.) The string suspended between the two parallel blue lines is infinitely heavy in this limit. So the 5d description suggests that the 6d $SU(2)$ theory at $n_{\mathbf{2}} = 4$ is given by $U(1)_g(\subset SU(2)_g)$ gauging of three factors, where two of them take the form of Fig. 4.13(a), and one takes the form of Fig. 4.13(b). Upon a suitable $SL(2,\mathbb{Z})$ transformation, Fig. 4.13(a) is the standard 5-brane web for the 5d $\mathcal{N} = 1$ pure $SU(2)$ theory. Similarly, Fig. 4.13(b) describes the 5d '$SU(1)$ theory.' The $SU(1)$ theory simply refers to the brane configuration of Fig. 4.13(b), not containing an interacting 5d SCFT. This sector will be void. So we shall take a suitable 5d scaling limit of the elliptic genera of 6d $SU(2)$ theory at $n_{\mathbf{2}} = 4$, and find the parameter map which exhibits two copies of 5d pure $SU(2)$ theories.

The 5d $SU(2)$ theory's BPS spectrum can be computed from its instanton

| $k \setminus n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 0 | $-2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | $-2$ | $-4$ | $-6$ | $-8$ | $-10$ | $-12$ | $-14$ |
| 2 | 0 | 0 | $-6$ | $-32$ | $-110$ | $-288$ | $-644$ |
| 3 | 0 | 0 | 0 | $-8$ | $-110$ | $-756$ | $-3556$ |
| 4 | 0 | 0 | 0 | 0 | $-10$ | $-288$ | $-3556$ |

Table 4.5: BPS spectrum of 5d $\mathcal{N}=1$ pure $SU(2)$ theory

partition function. It contains two fugacities, $Q$ for the instanton number, and $W$ for the $SU(2)$ electric charge in the Coulomb branch. It also contains Omega deformation parameters $\epsilon_{1,2}$. Here, we only consider the unrefined single particle spectrum, defined as follows. The partition function $Z^{SU(2)}(Q, W, \epsilon_{1,2})$ is written as $Z^{SU(2)} = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(Q^n, W^n, n\epsilon_{1,2})\right]$, where $f$ is the single particle index. Then one considers the limit

$$\lim_{\epsilon_{1,2}\to 0}\left[\left(2\sinh\frac{\epsilon_{1,2}}{2}\right) f(Q, W, \epsilon_{1,2})\right] \equiv f_{\rm rel}(Q, W) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} Q^k W^{2n} N_{k,n}\ . \tag{4.3.23}$$

The subscript ‘rel’ denotes the relative degrees of freedom of the bound states, as we divided the contribution $\frac{1}{4\sinh\frac{\epsilon_1}{2}\sinh\frac{\epsilon_2}{2}}$ from the center-of-mass degrees of freedom. We list some known results of $N_{k,n}$ in Table 4.5. The states at $k=0$, $n=1$ come from the perturbative partition function, from a massive 5d vector multiplet of W-boson. We would like to identify two copies of Table 4.5, by taking a 5d scaling limit of the elliptic genus for the instanton strings of 6d $SU(2)$ theory at $n_{\mathbf{2}}=4$. The elliptic genus can be obtained as a special case of (4.3.16) at $k_2 = k_3 = 0$.

After some trial-and-errors, we find it useful to expand the 6d index as

$$\begin{aligned} f_{\rm rel}(v,q,w,y_{1,2,3}) &= \sum_{n=0}^{\infty} e^{-nv} f_n(q,w,y_{1,2,3}) \\ &= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty} e^{-nv}\left(\frac{q}{w^2 y_3}\right)^p \left(w y_3^{-\frac{1}{2}}\right)^m N_{n,p,m}(y_{1,2,3}) \ , \end{aligned} \tag{4.3.24}$$

where $f_{\rm rel}$ is defined in the completely same manner as (4.3.23). $w$ is exponential of the $SU(2)$ Coulomb VEV, and $y_i \equiv e^{-v_i}$. We take the scaling limit $q \sim y_3^2 \sim w^4 \to 0$, with $v, y_1, y_2$ fixed. Note that $q \sim y_3^2$ is compatible with the scaling rules we already found, based on affine $SO(7)$ structure. The nonzero terms in this limit are listed in Tables 6,7,8,9, for $n \leq 4$. All terms except $N_{1,0,1}$ are finite in this limit. The two terms in $N_{1,0,1} \sim y_3^{-\frac{1}{2}}$ are divergent in the scaling limit. This implies the following situation. Suppose that we reduce $q \sim y_3^2 \sim w^4$, maintaining their ratios finite. Reducing $q$ physically means reducing the radius $R$ of $S^1$. When $e^{-v} w y_3^{-1} = 1$ or $e^{-v} w (y_1 y_2 y_3)^{-1} = 1$, the two terms in $N_{1,0,1}$ becomes 1, respectively. This means that the two states labeled by these terms become massless, causing a phase transition. Each term contributes $+1$ to the index, implying that $N_{1,0,1}$ comes from two hypermultiplets. Massless hypermultiplets cause flop phase transitions. Since the hypermultiplet's central charge changes sign after the transition, one should get $e^v w^{-1} y_3 (1 + y_1 y_2)$ after the two phase transitions. As we further reduce $q \sim y_3^2 \sim w^4$ to zero after the phase transitions, these two terms vanish, and we are left with the remaining finite numbers in the tables. One can then show that the remaining numbers in the tables are two copies of Table 4.5. Namely, one finds

$$
\begin{aligned}
f_{\rm rel} \;\; \to \;\; & -2e^{-v}\left[1+\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}\right]-4e^{-2v}\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}-6e^{-3v}\left[\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}+\left(\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}\right)^2\right] \\
& -e^{-4v}\left[8\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}+32\left(\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}\right)^2+8\left(\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}\right)^3\right]-\cdots \qquad (4.3.25) \\
& -2\frac{\mathcal{W}^2e^{-v}}{y_2}\left[1+\frac{y_2}{y_1}\right]-4\left(\frac{\mathcal{W}^2e^{-v}}{y_2}\right)^2\cdot\frac{y_2}{y_1}-6\left(\frac{\mathcal{W}^2e^{-v}}{y_2}\right)^3\left[\frac{y_2}{y_1}+\frac{y_2^2}{y_1^2}\right] \\
& -\left(\frac{\mathcal{W}^2e^{-v}}{y_2}\right)^4\left[8\frac{y_2}{y_1}+32\frac{y_2^2}{y_1^2}+8\frac{y_2^3}{y_1^3}\right]-\cdots\;,
\end{aligned}
$$

where $\mathcal{Q}\equiv\frac{q}{w^2y_3}$, $\mathcal{W}\equiv wy_3^{-\frac{1}{2}}$. The first two lines yield a 5d pure $SU(2)$ index, with the identification of Kahler parameters

$$
Q_1=\frac{\mathcal{Q}\mathcal{W}^2}{y_1y_2}=\frac{q}{y_1y_2y_3^2}(\equiv t_2)\;,\quad W_1^2=e^{-v}\;. \qquad (4.3.26)
$$

The last two lines yield another copy of 5d $SU(2)$ index, with parameters

$$
Q_2=\frac{y_2}{y_1}(\equiv t_1)\;,\quad W_2^2=\frac{\mathcal{W}^2e^{-v}}{y_2}=\frac{w^2e^{-v}}{y_2y_3}\;. \qquad (4.3.27)
$$

Note that the identifications of $Q_1,Q_2$ are consistent with our previous findings based on affine $SO(7)$ structure. This identifies the parameters (2), (7) of Fig. 4.11, and similarly (4), (9).

<u>$SO(7)$ **subsector:**</u> We now consider another subsector with $k_1=k_3=0,k_2\neq 0$. We start from the elliptic genus of the $SO(7)$ instanton strings at $n_{\mathbf{8}}=2$, studied in section 2. In the 5d scaling limit, e.g. at $k_2=1$, we found the following exact factorization,

$$
\begin{aligned}
f_{\rm rel} \;\; = \;\; & -U\left[\frac{1+Q_2}{W_2\tilde{W}_2(1-Q_2)^2}+(2\to 1)\right] \qquad (4.3.28) \\
& -2U^2\left[\frac{3Q_2^2+4Q_2^3+3Q_2^4}{(W_2\tilde{W}_2)^2(1-Q_2)^4(1-Q_2^2)^2}+(2\to 1)\right] \\
& -U^3\left[\frac{Q_2^3(27+70Q_2+119Q_2^2+119Q_2^3+70Q_2^4+27Q_2^5)}{(W_2\tilde{W}_2)^3(1-Q_2)^8(1-Q_2^3)^2}+(2\to 1)\right]+\cdots
\end{aligned}
$$

| $p \setminus m$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | $-2$ | $y_3^{-\frac{1}{2}}(1+\frac{1}{y_1 y_2})$ | $-2(\frac{1}{y_1}+\frac{1}{y_2})$ | 0 |
| 1 | 0 | 0 | $-\frac{2}{y_1 y_2}$ | 0 |
| 2 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 |

Table 4.6: $N_{1,p,m}$ in the scaling limit

| $p \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | $-\frac{4}{y_1 y_2}$ | 0 |
| 1 | 0 | 0 | $-\frac{4}{y_1 y_2}$ | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4.7: $N_{2,p,m}$ in the limit

| $p \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{6}{y_1 y_2}(\frac{1}{y_1}+\frac{1}{y_2})$ |
| 1 | 0 | 0 | $-\frac{6}{y_1 y_2}$ | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | $-\frac{6}{y_1^2 y_2^2}$ | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4.8: $N_{3,p,m}$ in the limit

| $p \setminus m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{1}{y_1 y_2}(\frac{8}{y_1^2}+\frac{8}{y_2^2}+\frac{32}{y_1 y_2})$ |
| 1 | 0 | 0 | $-\frac{8}{y_1 y_2}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | $-\frac{32}{y_1^2 y_2^2}$ | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | $-\frac{8}{y_1^3 y_2^3}$ | 0 | 0 |

Table 4.9: $N_{4,p,m}$ in the limit

where $U \equiv e^{-u-\frac{v+\tilde{v}}{2}} q^{-\frac{1}{2}} y_1 y_2 y_3^2$. So $f_{\text{rel}}$ decomposes into two factors. To have such a factorization, one should scale $e^{-u} \to \infty$ so that $e^{-u} q^{-1/2} y_3^2 \sim e^{-u} q^{1/2}$ is finite, which guarantees that $U$ is finite. Here, one can show that each factor takes the form of the instanton partition function for the 5d $\tilde{E}_1$ SCFT, upon identifying $U\left(Q_i^{1/2} W_i \tilde{W}_i\right)^{-1}$ as the instanton number fugacity and $Q_i$ as the electric charge fugacity (Coulomb VEV), for $i = 1, 2$ respectively. To understand

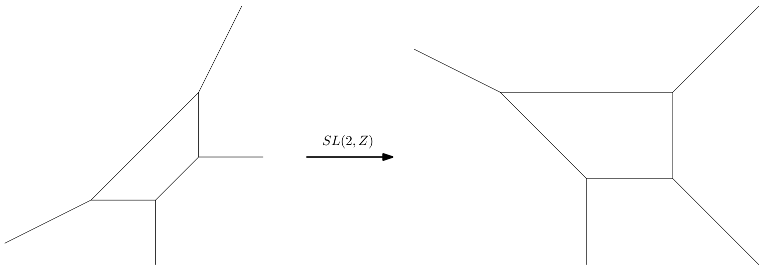



Figure 4.14: Brane web for the 5d $\tilde{E}_1$ SCFT

this from the brane web description, we take the 5d factorization limit $q \sim y_3^2 \to 0$, and also consider the limit $v, \tilde{v} \to \infty$ to realize the sector with $k_1 = k_3 = 0$. One finds that Fig. 4.11(b) decomposes into two $SU(1)$ theories in this limit since $(11) \to 0$, thus void. Each factor of Fig. 4.11(a) becomes the left side of Fig. 4.14, since $(2), (4), (7), (9) \to 0$. After an $SL(2, \mathbb{Z})$ transformation, it becomes the right side of Fig. 4.14. This is the standard brane configuration for the 5d $\tilde{E}_1$ theory [59]. It is the 5d $U(2)$ theory at Chern-Simons level 1. From these studies, one can identify the Kahler parameters (3), (8) of Fig. 4.11. Note that in (4.3.28), the leading term at $U^1$ order is $\frac{U}{W_i \tilde{W}_i}$ (with $i = 1, 2$) for the two 5d $\tilde{E}_1$ factors. This is the Kahler parameter for the bottom horizontal line on the right side of Fig. 4.14, since the leading BPS states come from the strings stretched along this line. So one finds $(3) = \frac{U}{W_1 \tilde{W}_1} = e^{-u} \frac{y_1 y_2 y_3^2}{\sqrt{q}}$, $(8) = \frac{U}{W_2 \tilde{W}_2} = e^{-u} \frac{y_1 y_2^2 y_3^3}{w \tilde{w} \sqrt{q}}$, which were already shown in Fig. 4.11. Once we know (3) and (8), one can determine (11) from the gluing condition $(11) = (2)(3)^2(4) = (7)(8)^2(9)$, again already shown in Fig. 4.11. Thus we fixed all Kahler parameters of Fig. 4.11 in terms of our 6d fugacities.

We have in fact made a nontrivial test of our elliptic genera of section 2, for the $SO(7)$ instanton strings at $n_{\mathbf{8}} = 2$, using the 5-brane web description, from (4.3.28). Although apparently we tested the elliptic genera in a 5d factorizing limit, this is different from the tests made in section 3. This is because the '5d limit' here scales other massive parameters and keeps a different slice of BPS states in its zero momentum sector. Indeed, using the original 6d variables, (4.3.28) is a nontrivial series in $Q_1 = \frac{q}{y_1 y_2 y_3^2} \sim q$, acquiring contributions from the 6d KK tower. So this provides an independent nontrivial test of our results in section 2.

**More general sectors:** We shall continue to study the scaling limit of the elliptic genera for more general winding sectors, at $(k_1, k_2, k_3) = (1,1,0), (1,2,0), (1,1,1), (1,2,1)$.

In the first three sectors, Fig. 11(b) factorizes to two '5d $SU(1)$' factors which are void, as these sectors are realized by $(4), (9), (11) \to 0$ for $(k_1, k_2, k_3) = (1,1,0), (1,2,0)$ and $(11) \to 0$ for $(k_1, k_2, k_3) = (1,1,1)$. So we expect the factorization of the single particle index into two identical pieces, each representing a non-Lagrangian 5d SCFT engineered by Fig. 11(a) in a particular limit. In all cases, we find exact factorizations of $f_{\rm rel}$ into two functions of identical form, as follows:

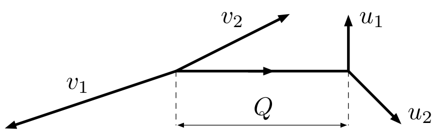
$$
\begin{aligned}
(1,1,0): \quad f_{\rm rel} &= e^{-v-u}\,\frac{y_1 y_2 y_3^2}{\sqrt{q}} \cdot \frac{(1+q/y_1 y_2 y_3^2)^2}{(1-q/y_1 y_2 y_3^2)^2} + e^{-v-u}\,\frac{w y_1 y_2 y_3^2}{\tilde{w}\sqrt{q}} \cdot \frac{(1+y_2/y_1)^2}{(1-y_2/y_1)^2} \\
(1,2,0): \quad f_{\rm rel} &= e^{-v-2u}\,\frac{y_1^2 y_2^2 y_3^4}{q} \cdot \frac{-10\,(q/y_1 y_2 y_3^2)^2\,(1+q/y_1 y_2 y_3^2)}{(1-q/y_1 y_2 y_3^2)^6} \\
&\quad + e^{-v-2u}\,\frac{y_1^2 y_2^3 y_3^5}{\tilde{w}^2 q} \cdot \frac{-10\,(y_2/y_1)^2\,(1+y_2/y_1)}{(1-y_2/y_1)^6} \\
(1,1,1): \quad f_{\rm rel} &= e^{-v-u-\tilde{v}}\,\frac{y_1 y_2 y_3^2}{\sqrt{q}}\,\frac{(1+q/y_1 y_2 y_3^2)^3}{(1-q/y_1 y_2 y_3^2)^2} + e^{-v-u-\tilde{v}}\,\frac{w\tilde{w} y_1 y_3}{\sqrt{q}}\,\frac{(1+y_2/y_1)^3}{(1-y_2/y_1)^2}.
\end{aligned}
\tag{4.3.29}
$$

Each term is a product of the prefactor $(2)^{k_1}(3)^{k_2}(4)^{k_3}$ or $(7)^{k_1}(8)^{k_2}(9)^{k_3}$ and a function of Kähler parameter $(1) = (5)$ or $(6) = (10)$, respectively. To test these results, extracted from the elliptic genera in Section 4.1, we shall independently do the topological vertex calculus for the 5d SCFT of Fig. 11(a).

The topological vertex [60] computes all genus topological string amplitudes, which is equivalent to the logarithm of the 5d Nekrasov partition function on Omega-deformed $\mathbf{R}^4 \times S^1$ [61]. Here we refer to [50, 51] for its detailed description. We select an orientation of every edge in the 5-brane web. Each internal edge is associated with a Young diagram. We also assign an empty Young diagram to every external edge. The 5d partition function is given by a sum over all combinations of Young diagrams. The summand is a product of factors coming from every edge and vertex. We turn off $\epsilon_+ = 0$ to simplify the formulae. When all three edges are outgoing from a given vertex, the vertex factor is given by (where $u = e^{-\epsilon_-}$, $||\mu||^2 = \sum_i \mu_i^2$)

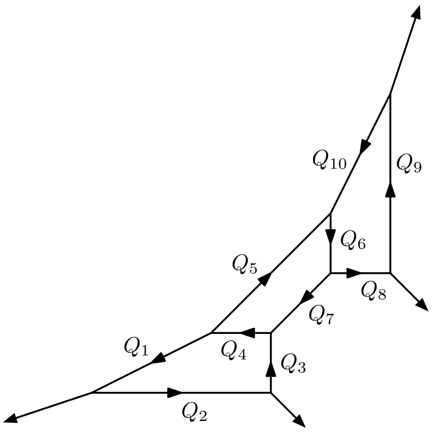
$$C_{\lambda\mu\nu}(u) = u^{\frac{||\mu||^2+||\nu||^2-||\mu^t||^2}{2}} \prod_{s\in\nu}(1-u^{l_\nu(s)+a_\nu(s)+1})^{-1} \cdot \sum_\eta s_{\lambda^t/\eta}(u^{-\rho}u^{-\nu}) s_{\mu/\eta}(u^{-\rho}u^{-\nu^t}) \,. \tag{4.3.30}$$

$\lambda, \mu, \nu$ are Young diagrams associated to the edges. For an incoming edge, the assigned Young diagram should be transposed. The skew-Schur function $s_{\lambda/\eta}(\mathbf{x})$ depends on a possibly infinite vector $\mathbf{x}$, which in above is $u^{-\rho}u^{-\nu} \equiv (u^{\frac{1}{2}-\nu_1}, u^{\frac{3}{2}-\nu_2}, u^{\frac{5}{2}-\nu_3}, \cdots)$. The functions $l_\nu(s)$ and $a_\nu(s)$ are defined by $l_\nu(s) = \nu_i - j$ and $a_\nu(s) = \nu^t_j - i$, where $i, j$ represent the horizontal and vertical positions of the box $s$ from the upper-left corner of $\nu$. It is known that $C_{\lambda\mu\nu}(u)$ is invariant under the cyclic permutation of $\lambda, \mu, \nu$ using Schur function identities [60]. An internal edge glues a pair of vertices by multiplying the edge factor and summing over the assigned Young diagram. Denoting its Kähler parameter by $Q$, the edge factor is given by




$$= \quad (-Q)^{|\nu|} f_\nu(u)^n \tag{4.3.31}$$

where $f_\nu(u) = (-1)^{|\nu|} u^{\frac{||\nu^t||^2-||\nu||^2}{2}}$ and $n = \det(u_1, v_1)$. Applying these rules, one obtains the following partition function of 5d SCFT engineered from the brane web of Fig. 11(a),




$$
= \quad \begin{aligned} &\textstyle\sum_{\nu_1,\cdots,\nu_{10}} \prod_{i=1}^{10} (-Q_i)^{|\nu_i|} \\ &\times f_{\nu_1} f_{\nu_2}^3 f_{\nu_3} f_{\nu_4}^{-1} f_{\nu_5}^{-2} f_{\nu_6}^{-1} f_{\nu_8} f_{\nu_9}^3 f_{\nu_{10}} \\ &\times C_{\nu_1^t \nu_2 \phi} C_{\nu_2^t \nu_3 \phi} C_{\nu_3^t \nu_4 \nu_7^t} C_{\nu_4^t \nu_1 \nu_5} \\ &\times C_{\nu_5^t \nu_{10}^t \nu_6} C_{\nu_6^t \nu_8 \nu_7} C_{\nu_8^t \nu_9 \phi} C_{\nu_9^t \nu_{10} \phi} \end{aligned} \quad . \tag{4.3.32}
$$

$Q_1, \cdots, Q_{10}$ are identified with the Kähler parameters in Fig. 11(a) as

$$
Q_1 = Q_3 = \alpha,\ Q_4 = Q_6 = \beta,\ Q_7 = \gamma,\ Q_8 = Q_{10} = \delta,\ Q_2 = \alpha^2\beta,\ Q_5 = \beta\gamma,\ Q_9 = \beta\delta^2 \tag{4.3.33}
$$

where $(\alpha, \beta, \gamma, \delta) = (e^{-v}, \frac{q}{y_1 y_2 y_3^2}, e^{-u}\frac{y_1 y_2 y_3^2}{\sqrt{q}}, e^{-\tilde{v}})$ or $(\frac{w^2 e^{-v}}{y_2 y_3}, \frac{y_2}{y_1}, e^{-u}\frac{y_1 y_2^2 y_3^3}{w\tilde{w}\sqrt{q}}, \frac{\tilde{w}^2 e^{-\tilde{v}}}{y_2 y_3})$ respectively. To derive the single particle spectrum of $(k_1, k_2, k_3)$ sector, we perform the sum (4.3.32) over Young diagrams until $|\nu_1| + 2|\nu_2| + |\nu_3| \leq k_1$, $|\nu_5| + |\nu_7| \leq k_2$, $|\nu_8| + 2|\nu_9| + |\nu_{10}| \leq k_3$ and take the Plethystic logarithm. To compare with (4.3.29), we further multiply $-(2\sinh\frac{\epsilon_-}{2})^2$ on it and take the limit $\epsilon_- \to 0$. After these manipulations, one obtains

$$
\begin{aligned}
(1,1,0):&\quad f_{\text{top}} = \alpha\gamma \cdot \big(1 + 4\beta + 8\beta^2 + 12\beta^3 + 16\beta^4 + 20\beta^5 + \mathcal{O}(\beta^6)\big) \\
(1,2,0):&\quad f_{\text{top}} = \alpha\gamma^2 \cdot \big(-10\beta^2 - 70\beta^3 - 270\beta^4 - 770\beta^5 + \mathcal{O}(\beta^6)\big) \\
(1,1,1):&\quad f_{\text{top}} = \alpha\gamma\delta \cdot \big(1 + 5\beta + 12\beta^2 + 20\beta^3 + 28\beta^4 + 36\beta^5 + \mathcal{O}(\beta^6)\big)\,.
\end{aligned} \tag{4.3.34}
$$

These agree with $f_{\text{rel}}$ in (4.3.29), testing our elliptic genera in Section 4.1.

We finally consider the $(1,2,1)$ sector. $f_{\text{rel}}$ in the factorization limit is given by

$$f_{\text{rel}} = e^{-v-2u-\tilde{v}}\frac{y_1^2y_2^2y_3^4}{q}\cdot\left[\frac{-2y_1y_2}{(1-y_1y_2)^2}+\left(\frac{-2k\left(k^4+4k^3+30k^2+4k+1\right)}{(k-1)^6}\right)\Big|_{k=\frac{y_2}{y_1}}\right.$$
$$\left.+\left(\frac{-2k\left(k^4+4k^3+30k^2+4k+1\right)}{(k-1)^6}\right)\Big|_{k=\frac{q}{y_1y_2y_3^2}}+(-2)\right]. \tag{4.3.35}$$

The common prefactor $e^{-v-2u-\tilde{v}}y_1^2y_2^2y_3^4/q$ is $(2)(3)^2(4)=(7)(8)^2(9)=(11)$. The first term agrees with the 1 instanton partition function of 5d pure $SU(2)$ gauge theory, if we identify $\sqrt{y_1y_2}$ as the fugacity of the $SU(2)$ electric charge. It belongs to the 5d $E_1$ SCFT of Fig. 11(b). The next two terms take the same functional form, respecting the $\mathbb{Z}_2$ symmetry of the two factors. To test this function, we performed the topological vertex calculus for (4.3.32). We first sum over all Young diagrams with $|\nu_1|+2|\nu_2|+|\nu_3|\leq 1$, $|\nu_5|+|\nu_7|\leq 2$, $|\nu_8|+2|\nu_9|+|\nu_{10}|\leq 1$ and take the Plethystic logarithm. We then subtract the extra factor $\alpha\gamma^2\delta(2\sinh\frac{\epsilon_-}{2})^{-2}$ that arises because the strings can propagate along the parallel 5-branes [50, 51]. Dividing out the center-of-mass factor $-(2\sinh\frac{\epsilon_-}{2})^{-2}$ and turning off $\epsilon_-\to 0$, the topological string partition function becomes

$$(1,2,1):\quad f_{\text{top}}=\alpha\gamma^2\delta\cdot\left(-2\beta-20\beta^2-150\beta^3-648\beta^4-2010\beta^5+\mathcal{O}(\beta^6)\right). \tag{4.3.36}$$

It agrees with the second and third terms of $f_{\text{rel}}$ in (4.3.35). The final $(-2)$ comes from the perturbative $SU(2)_g$ vector multiplet. Again, this result gives a non-trivial independent test of our elliptic genera in Section 4.1.

### 4.3.3 $3,2$ and $3,2,2$: $G_2\times SU(2)$ gauge group

We construct 2d quivers for the strings of other 6d SCFTs in Table 4.4. The tests we can provide about them are weak (e.g. anomalies). We keep the presentations rather brief.

<u>$3, 2, 2$ **SCFT strings:**</u> The strategy is similar to that of section 4.1. We first consider the limits in which all except one gauge symmetry are ungauged in 6d, and take three factors of ADHM(-like) quivers. We then combine these quivers by locking certain symmetries, and introducing bi-fundamental matters of the form of (4.3.2). To be more precise, we have no 6d gauge group associated with the '2' node on the right. Although the notion of ungauging is absent for this node, we can still take the tensor VEV associated with this node to infinity. Whenever a node has a 6d gauge group, its inverse coupling is proportional to the tensor VEV $\langle\Phi\rangle$, so taking $\langle\Phi\rangle \to \infty$ ungauges the symmetry.

If one takes all tensor VEVs to infinity except the '3' node, one obtains the 6d $G_2$ theory at $n_{\mathbf{7}} = 1$. This is because the 6d matter in $\frac{1}{2}(\mathbf{7}, \mathbf{2})$ behaves like one full hypermultiplet in $\mathbf{7}$, while $\frac{1}{2}(\mathbf{1}, \mathbf{2})$ is neutral in $G_2$ and invisible in the gauge dynamics. So with a $G_2$ theory at $n_{\mathbf{7}} = 1$, its $k_1$ $G_2$ instanton strings are described by the 2d $U(k_1)$ gauge theory explained in section 2.2, with fields given by (4.1.49), (4.1.50), (4.1.51) at $n_{\mathbf{7}} = 1$. The ungauged $SU(2) \sim Sp(1)$ acts as the flavor symmetry of the 6d hypermultiplet. In the ADHM-like quiver at general $n_{\mathbf{7}}$, one may have as big as $U(2n_{\mathbf{7}})$ flavor symmetry which rotates Fermi multiplets. But the coupling to bulk fields only allowed $U(n_{\mathbf{7}})$ part, which we further expected to enhance to $Sp(n_{\mathbf{7}})$. This is similar to the flavor symmetries of $SO(7)$ ADHM-like theory at $n_{\mathbf{8}} \neq 0$. In the current context, again like the $2, 3, 2$ quiver, we should couple the system to different bulk fields. At $n_{\mathbf{7}} = 1$, one can classically have as big as $U(2n_{\mathbf{7}}) \to U(2)$ flavor symmetry. We restrict it to $SU(2)$ which rotates $\Psi, \tilde{\Psi}^\dagger$ of (4.1.51) as a doublet. Also, as explained in section 2.2, only $SU(3) \subset G_2$ is visible in this quiver. More formally, it will be convenient to regard the fields $q_i, \tilde{q}^i, \phi_i, \phi_4$ as transforming in $SU(3) \times SU(1) \subset$

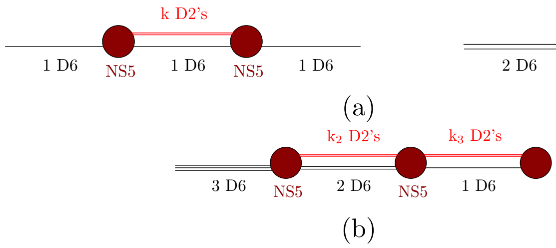



Figure 4.15: Brane configurations for: (a) 6d $(2,0)$ SCFT of type $A_1$; (b) 6d $3,2,2$ SCFT in the limit with ungauged $G_2$

$SU(4)$.

When $G_2$ is ungauged and the tensor VEV for the right '2' node is sent to infinity, we have 6d $SU(2)$ theory at $n_{\mathbf{2}} = 4$. Its ADHM quiver is explained around (4.3.1). In this limit, $G_2$ is enhanced to $SO(7)$ flavor symmetry rotating the four hypermultiplets in $\mathbf{8}$ of $SO(7)$, but only $SU(4) \subset SO(7)$ is visible in the UV ADHM, as explained in section 4.1. $SO(7)$ will later be broken to $G_2$ by gauging. In our ADHM-like quiver, which only sees $SU(3) \subset G_2$, $SU(4)$ will be broken to $SU(3) \times SU(1)$, locked with the $G_2$ ADHM of the previous paragraph.

We finally ungauge $G_2 \times SU(2)$, leaving one tensor VEV for the right '2' node finite. One then obtains the 6d $\mathcal{N} = (2,0)$ SCFT of $A_1$ type, geometrically engineered on the $O(-2) \to \mathbb{P}^1$ base with no associated gauge group. Although the strings of this SCFT in the tensor branch lacks the instanton string interpretation, one still knows the UV 2d gauge theory description [14]. For $k$ strings, this is a $U(k)$ gauge theory. The 2d fields are given by

$$\begin{aligned} (A_\mu, \lambda_0, \lambda) &\ : \ \text{vector mutiplet in } (\mathbf{adj}, 0) \\ q_{\dot\alpha} = (q, \tilde{q}^\dagger) &\ : \ \text{hypermultiplet in } (\mathbf{k}, -1) \\ a_{\alpha\dot\beta} \sim (a, \tilde{a}^\dagger) &\ : \ \text{hypermultiplet in } (\mathbf{adj}, 0) \\ \Psi_a &\ : \ \text{Fermi multiplet in } (\mathbf{k}, 0)\ , \end{aligned} \tag{4.3.37}$$

where $a = 1, 2$. We showed the representation and charge of the classical symmetry $U(k) \times U(1)$, where one should further restrict $U(1) \to SU(1)$ due to mixed anomaly. This formally takes the form of the ADHM instanton strings of '6d $SU(1)$ theory' with two charged quarks. The $SU(2)_F$ flavor symmetry which rotates $\Psi_a$ is identified with the enlarged R-symmetry group of the 6d $(2, 0)$ theory. Namely, we expect that $SU(2)_R$ of 6d $(1, 0)$ SCFT enhances to $SO(5)_R$. In the tensor branch, this is broken to $SO(4) \sim SU(2)_R \times SU(2)_L$, where the latter $SU(2)_L$ is realized as $SU(2)_F$ in the 2d quiver. The 6d $A_1$ $(2, 0)$ theory and the above 2d gauge theory admit D-brane engineerings. Using D2-D6-NS5, one can use either of Fig. 4.15(a), in IIA or massive IIA string theory [62, 63].

Before fully combining the three ADHM(-like) quivers, we note that the combination of two '2' nodes (with $G_2$ ungauged) is dictated by a D-brane setting. This is given by the brane configuration of Fig. 4.15(b) in the massive IIA theory. The 2d quiver is given by Fig. 4.16 at $k_1 = 0$. The quiver and the brane system only has manifest $SU(3) \times SU(2) \times U(1)$ symmetry, where the last $U(1)$ is a combination of three overall $U(1)$'s in $U(3) \times U(2) \times U(1)$ which survive the mixed anomaly cancelation with $U(k_2) \times U(k_3)$. More precisely, taking the overall $U(1)$ generators $Q_i$ for $SU(i)$, $i = 1, 2, 3$, only $Q_1 + Q_2 + Q_3$ is free of the mixed anomaly. (This $U(1)$ is not shown in Fig. 4.16, as it will be irrelevant generally at $k_1, k_2, k_3 \neq 0$.) One can see that the 2d quiver exhibits $SU(3) \times U(1) \to SO(7)$ symmetry enhancement, say by studying the elliptic genera. This should be the case since one has 6d $SU(2)$ theory at $n_{\mathbf{2}} = 4$. Just to be sure, we tested the $SO(7)$ enhancement of the elliptic genus at $k_2 = k_3 = 1$.

Now we keep $k_1 \neq 0$, with $G_2$ gauged. In our UV GLSM, we can only see $SU(3) \subset G_2$, which we lock with the $SU(3)$ symmetry of the quiver in the

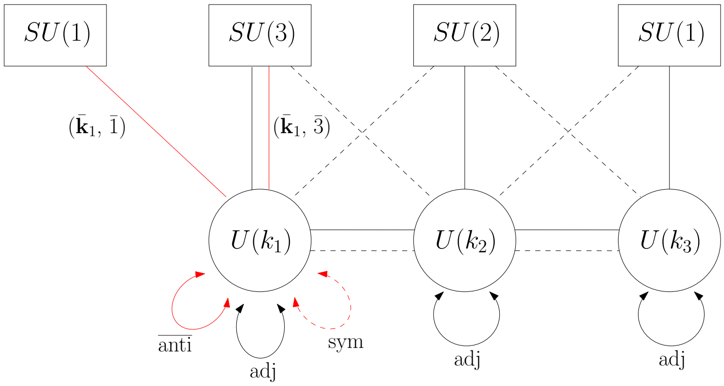



Figure 4.16: 2d quiver for the strings of 6d 3, 2, 2 SCFT

previous paragraph. The resulting $U(k_1) \times U(k_2) \times U(k_3)$ quiver is given by Fig. 4.16. The potentials can be written down in a similar manner as the 2, 3, 2 quiver of section 4.1. We skip the details here.

As a small test of our quiver, we compute the 2d anomalies. We first compute it from inflow. The Green-Schwarz part of the 6d anomaly 8-form is given by $I_{GS} = \frac{1}{2}\Omega^{ij} I_i I_j$ with

$$I_i = \begin{pmatrix} \frac{1}{4}\mathrm{Tr}(F_{G_2}^2) + \alpha_1 c_2(R) + \alpha_2 p_1(T) \\ \frac{1}{4}\mathrm{Tr}(F_{SU(2)}^2) + \beta_1 c_2(R) + \beta_2 p_1(T) \\ \gamma_1 c_2(R) + \gamma_2 p_1(T) \end{pmatrix}, \quad \Omega^{ij} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{4.3.38}$$

where $(\alpha_1, \beta_1, \gamma_1) = (\frac{17}{7}, \frac{23}{7}, \frac{15}{7})$, $(\alpha_2, \beta_2, \gamma_2) = (\frac{3}{28}, \frac{1}{14}, \frac{1}{28})$. We explain how to get this result. [42] uses two methods to compute $I_{GS}$. One is applicable when all nodes have gauge symmetries. In this case, one demands that $I_{GS}$ cancels all terms in $I_{\text{1-loop}}$ containing dynamical fields. This is the method we used so far in this paper. When some nodes do not have gauge symmetries, this method alone cannot completely determine $I_{GS}$. We use the following strategy to compute (4.3.38). Firstly, we compute the 1-loop anomaly containing the dynamical $G_2 \times SU(2)$ gauge fields, and demand that this part is completely canceled by $I_1, I_2$ part of $I_{GS}$ [41]. Then one obtains $I_i$ of the form (4.3.38), where the six coefficients $\alpha_{1,2}, \beta_{1,2}, \gamma_{1,2}$ are constrained only by the following four equations,

$$3\alpha_1 - \beta_1 = 4 \ , \ \ 3\alpha_2 - \beta_2 = \frac{1}{4} \ , \ \ 2\beta_1 - \alpha_1 - \gamma_1 = 2 \ , \ \ 2\beta_2 - \alpha_2 - \gamma_2 = 0 \ . \quad (4.3.39)$$

To further constrain them, we consider the limit in which two tensor VEVs are sent to infinity so that $G_2 \times SU(2)$ are ungauged. In this limit, we can use the known expression for $I_{GS}$ for the $A_1$ $(2,0)$ theory in the tensor branch [41, 42],

$$I_{GS} = \frac{1}{2}\Omega\left(\frac{1}{2}(c_2(R) - c_2(L))\right)^2 \ , \ \ \Omega = 2 \ , \quad (4.3.40)$$

with enhanced $SU(2)_R \times SU(2)_L = SO(4) \subset SO(5)$ R-symmetry. After taking this limit, we can set $\frac{1}{4}\mathrm{Tr}(F^2_{SU(2)}) = c_2(L)$ by identifying the ungauged $SU(2)$ with $SU(2)_L$. To take this limit, consider the vector kinetic terms proportional to $\mathcal{L}_{\mathrm{v}} \sim \Omega^{ij}\Phi_i \mathrm{Tr}(F_j^2) \equiv \Phi^i \mathrm{Tr}(F_i^2)$. We keep $\Phi^3 = 2\Phi_3 - \Phi_2$ finite, while taking $\Phi^1 = 3\Phi_1 - \Phi_2$ and $\Phi^2 = 2\Phi_2 - \Phi_1 - \Phi_3$ to $+\infty$, to ungauge $G_2 \times SU(2)$. To properly do so, note that the kinetic terms for $\Phi_i$ are proportional to $\mathcal{L}_{\mathrm{t}} \sim \Omega^{ij}\partial^\mu\Phi_i\partial_\mu\Phi_j$. This is diagonalized by taking, say, $\Phi_3 = a + \chi$, $\Phi_2 = 2a$, $\Phi_1 = b + \frac{2a}{3}$, since $\mathcal{L}_t \sim \frac{14}{3}(\partial a)^2 + 3(\partial b)^2 + 2(\partial\chi)^2$. So one holds the scalars $a, b$ very large and fixed, unaffected by the dynamical $\chi$ and its superpartner. More precisely, $a, b$ can be hold fixed, given by infinite constant plus a finite background function given by the background gauge fields. $\chi$ is a dynamical scalar associated with the right '2' node with normalization $\Omega = 2$. In this parameterization $\chi, a, b$ of tensor multiplet scalars, one can similarly show that the superpartners $H_a, H_b$ of $a, b$ can be consistently taken to be fixed background functions, unaffected by dynamical $\chi$ and its superpartner $H_\chi$. Now consider the equation of motion for $H_\chi$. The coupling between $B_\chi$

and the dynamical/background vector fields is given by

$$\Omega^{ij}B_i\wedge I_j\to B_\chi\Omega^{3i}I_i=B_\chi\wedge(2I_3-I_2)\ . \tag{4.3.41}$$

We used $\Omega^{ij}B_j=(\cdots,-B_\chi+\cdots,2B_\chi)$, where $\cdots$ depend on $B_a,B_b$, so that it depends on $B_\chi$ as $\Omega^{i3}B_\chi$. From the equation of motion for $B_\chi$, one obtains

$$d\star H_\chi=(\Omega^{33})^{-1}\Omega^{3i}I_i=I_3-\frac{1}{2}I_2\ . \tag{4.3.42}$$

By comparing this with (4.3.40), one obtains $I_3-\frac{1}{2}I_2=\frac{c_2(R)-c_2(L)}{2}$ with $c_2(L)=\frac{1}{4}\mathrm{Tr}(F^2_{SU(2)})$. This leads to two more equations for $\alpha_{1,2},\beta_{1,2},\gamma_{1,2}$,

$$2\gamma_1-\beta_1=1\ ,\quad 2\gamma_2-\beta_2=0\ . \tag{4.3.43}$$

The unique solution of (4.3.39), (4.3.43) is the one stated right below (4.3.38).[7]

This leads to the anomaly 4-form on the strings from inflow, (4.1.40), given by

$$I_4=\left(k_1k_2+k_2k_3-\frac{3}{2}k_1^2-k_2^2-k_3^2\right)\chi(T_4)+k_1(I_2-3I_1)+k_2(I_1+I_3-2I_2)+k_3(I_2-2I_3)\ . \tag{4.3.44}$$

This is computed from our 2d gauge theory as follows. We again decompose the anomaly into contributions $I_4^{(1)}$ from the $G_2$ ADHM-like quiver, $I_4^{(2)}$ from the middle '2' node (6d $SU(2)$ theory at $n_{\mathbf{2}}=4$), $I_4^{(3)}$ from the right '2' node, and $I_4^{\rm bif}=(k_1k_2+k_2k_3)\chi(T_4)$. $I_4^{(1)}$ and $I_4^{(2)}$ are given by (4.1.62), and (4.3.12) replacing $F_{SO(7)}\to F_{G_2}$. $I_4^{(3)}$ is given by eqn.(5.21) of [5] at $N=1$,

$$I_4^{(3)}=-\frac{k_3}{2}\mathrm{Tr}(F^2_{SU(1)})+\frac{k_3}{4}\mathrm{Tr}(F^2_{SU(2)})-k_3c_2(R)-k^2\chi(T_4)\ , \tag{4.3.45}$$

[7] In fact, expanding the arguments of this paragraph, one can compute $I_{GS}$ if one knows the Green-Schwarz anomalies of all individual rank 1 nodes before combining them. The general rule is as follows. Suppose that $I_{\rm GS}^{(i)}=\frac{1}{2}\Omega^{ii}(I_i)^2_{\rm single}$ (no sum of $i$) when only $i$'th node is kept. Then defining $I^i\equiv(\Omega^{ii})^{-1}(I_i)_{\rm single}$ (no sum of $i$), one finds $I_{\rm GS}=\frac{1}{2}(\Omega^{-1})_{ij}I^iI^j$. $I_i$ that we computed in (4.3.9) and (4.3.38) are given by $I_i=(\Omega^{-1})_{ij}I^j$.

where one should set $F_{SU(1)} = 0$. Adding $\sum_{i=1}^{3} I_4^{(i)} + I_4^{\text{bif}}$, one precisely reproduces (4.3.44).

3, 2 **SCFT strings:** This SCFT can be obtained from the previous 3, 2, 2 SCFT by taking the tensor VEV of the right '2' node to infinity. The corresponding 2d quiver for its strings can be obtained from our previous quiver for the 3, 2, 2 model, by taking $k_3 = 0$. All the discussions made for the 3, 2, 2 string quivers apply here as well.

# Chapter 5

# Instantons from blow-up

## 5.1 Instanton Counting from Blow-up

In this chapter, we introduce completely different method to compute instanton partition functions using blow-up. The essential idea of using the blow-up of $\mathbb{C}^2$ for instanton counting is that the gauge theory partition function for a 4d $\mathcal{N}=2$ (or 5d $\mathcal{N}=1$) theory on the blow-up of a point $\hat{\mathbb{C}}^2$ (or $\hat{\mathbb{C}}^2 \times S^1$) can be written in two different ways. This will allow us to write a recursion relation for the instanton partition function that can be solved rather easily [77–79, 90].

### 5.1.1 Blowup equation

**Localization on the blow-up $\hat{\mathbb{C}}^2$** One of the expressions for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ comes from the Coulomb branch localization, which results that $\hat{\mathcal{Z}}$ can be obtained by patching together the flat-space partition function $\mathcal{Z}$ [97].

The blow-up $\hat{\mathbb{C}}^2$ of the complex plane is constructed from $\mathbb{C}^2$ by replacing the origin with a compact 2-cycle $\mathbb{P}^1$. In particular, the geometry is identical to the total space of the line bundle of degree $(-1)$ over $\mathbb{P}^1$. One can parametrize $\mathcal{O}(-1) \to \mathbb{P}^1$ using the homogeneous coordinates $(z_0, z_1, z_2)$, satisfying the projective condition $(z_0, z_1, z_2) \sim (\lambda^{-1} z_0, \lambda^1 z_1, \lambda^1 z_2)$ for any $\lambda \in \mathbb{C}^*$, where the two-cycle $\mathbb{P}^1 \subset \hat{\mathbb{C}}^2$ corresponds to the locus $z_0 = 0$. We are interested in the $U(1)^2$ equivariant partition function, with the $U(1)^2$ action $V$ rotating the complex coordinates $(z_0, z_1, z_2)$ as follows:

$$(z_0, z_1, z_2) \mapsto (z_0, e^{\epsilon_1} z_1, e^{\epsilon_2} z_2). \tag{5.1.1}$$

Instantons are located at two fixed points of the $U(1)^2$ action, *i.e.*, the north/south poles of the $\mathbb{P}^1$, whose coordinates are $(z_0, z_1, z_2) = (0, 1, 0)$ and $(0, 0, 1)$. Around these fixed points, ($\mathbb{C}^*$-invariant) local coordinates are given by $(z_0 z_1, z_2/z_1)$ and $(z_0 z_2, z_1/z_2)$ respectively. The local weights under the $U(1)^2$ action $V$ near the fixed points are:

$$\begin{aligned} &(z_0 z_1,\ z_2/z_1) \mapsto (e^{\epsilon_1} z_0 z_1,\ e^{\epsilon_2 - \epsilon_1} z_2/z_1) && \text{(near the north pole)} \\ &(z_0 z_2,\ z_1/z_2) \mapsto (e^{\epsilon_2} z_0 z_2,\ e^{\epsilon_1 - \epsilon_2} z_1/z_2) && \text{(near the south pole)} \end{aligned} \tag{5.1.2}$$

The full partition function $\hat{\mathcal{Z}}$ on $\hat{\mathbb{C}}^2$, which includes both the perturbative and instanton contributions, can be obtained by performing the localization on the Coulomb branch. On the Coulomb branch, the gauge group is generically broken to $U(1)^r$ where $r$ is the rank of the gauge group. The $U(1)^r$ equivariant parameters $\vec{a}$ naturally appear in the partition function. One needs to sum over all distinct field configurations with zero-sized instantons located at the north and south poles. All the inequivalent configurations are labeled by the $r$-dimensional vector $\vec{k}$ of the first Chern numbers, corresponding to different flux configurations on the two-cycle $\mathbb{P}^1$. When the gauge group has $U(1)$ factor, we can turn on the external flux that can be supported on the $\mathbb{P}^1$. We assume there is no such a factor in the gauge group. Summing up, $\hat{\mathcal{Z}}$ can be expressed in terms of the partition function $\mathcal{Z}$ on $\mathbb{C}^2$ as [75, 81, 95–97]

$$\hat{\mathcal{Z}}(\vec{a}, \epsilon_1, \epsilon_2, q, \vec{m}) = \sum_{\vec{k}\in\Lambda} \mathcal{Z}^{(N)}(\vec{k})\mathcal{Z}^{(S)}(\vec{k}) \ , \tag{5.1.3}$$

where the flux sum is taken over the co-root lattice $\Lambda$ of the gauge algebra. Each factor represents the partition function localized at the $U(1)^2$ fixed points (north/south-poles of the $\mathbb{P}^1 \subset \hat{\mathbb{C}}^2$) given as

$$\begin{aligned} \mathcal{Z}^{(N)}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, q, \vec{m} - \frac{1}{2}\epsilon_1) \ , \\ \mathcal{Z}^{(S)}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, q, \vec{m} - \frac{1}{2}\epsilon_2) \ . \end{aligned} \tag{5.1.4}$$

In addition to the Coulomb branch parameters, the partition function depends on the Omega deformation parameters $\epsilon_1, \epsilon_2$ and also mass parameters $\vec{m}$. The instanton fugacity $q$ takes the following form: For a 4d theory, it is given as $q = e^{2\pi i\tau} = \Lambda^{b_0}$ where $\tau$ is the complexified gauge coupling and $\Lambda$ being the dynamical scale of the gauge theory. The exponent $b_0$ is the 1-loop beta function coefficient. For a 5d theory, it is also given by the exponentiated gauge coupling as $q = e^{-\frac{1}{g^2}} \equiv e^{-m_0}$. Notice that the Coulomb parameter $\vec{a}$ gets an appropriate shift at each fixed point $p$, induced by the non-trivial magnetic flux $\vec{k}$ on the blown-up $\mathbb{P}^1$, with the proportionality constant $H|_p$. The values of the moment map $H$ for the $U(1)^2$ action $V$, *i.e.*, $dH = \iota_V\omega$, at the north and south poles are given as

$$H|_{\text{NP}} = \epsilon_1 \text{ and } H|_{\text{SP}} = \epsilon_2. \tag{5.1.5}$$

The mass parameters also get shifted since the hypermultiplet mass is twisted by $SU(2)_R$, which makes the combination $m - \frac{\epsilon_1+\epsilon_2}{2}$ invariant at the fixed points.[1]

[1] One can instead use the shifted mass to simplify the formula involving mass. We use unshifted mass to match with the existing formulae in the literature.

**Partition function on $\hat{\mathbb{C}}^2$ vs $\mathbb{C}^2$** Another important fact for the partition function $\hat{\mathcal{Z}}$ on the blow-up $\hat{\mathbb{C}}^2$ is that $\hat{\mathcal{Z}}$ is actually identical to the flat-space partition function $\mathcal{Z}$ [72,77–79,81,82]. The blow-up $\hat{\mathbb{C}}^2$ is identical to $\mathbb{C}^2$ except for the origin, which is replaced by the blown-up sphere $\mathbb{P}^1$. Since the Nekrasov partition function gets contributions only from the small instantons localized at the fixed points of the $U(1)^2$ equivariant action $V$, the size of the divisor should not affect the partition function as we smoothly shrink it. So we expect that $\hat{\mathcal{Z}} = \mathcal{Z}$. This implies the following relation:

$$\mathcal{Z} = \hat{\mathcal{Z}} = \sum_{\vec{k}\in\Lambda} \mathcal{Z}^{(N)}(\vec{k})\mathcal{Z}^{(S)}(\vec{k}). \tag{5.1.6}$$

This blow-up identity can be thought of as a special case of more generalized orbifold partition functions [94, 95, 113]. For example, the Nekrasov partition function on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ can be computed in two different ways, one is via formula analogous to (5.1.3) by combining the contributions from two fixed points of the blown-up geometry $\mathcal{O}(-2) \to \mathbb{P}^1$. The other way is to compute the partition function at the orbifold point using the ADHM construction for the orbifolds. The Nekrasov partition function still remains the same as we blow up or down the singular point.[2] The only difference in our case is that we blow-up or down a non-singular point instead of a singular point.

**Correlation functions in 4d** The equation (5.1.6) itself is not enough to fix the partition function completely, since there are 3 unknown functions and only one relation. It turns out the necessary additional relations can be found from the insertion of non-trivial $\mathcal{Q}$-closed operators [77, 79] associated to the two-cycle on the blow-up.

[2]This simple picture does not necessarily hold when there are too many hypermultiplets, due to some subtle scheme dependence related to the wall-crossing [94].

In the 4d Donaldson-twisted theory, the $\mathcal{Q}$-invariant observable $\mathcal{O}_2$ associated to a two-cycle can be constructed by applying the topological descent procedure twice to the Casimir invariant $\mathcal{O}_0 = \mathrm{Tr}(\Phi^2)$ as [105]

$$\begin{aligned} 0 = \{\mathcal{Q}, \mathcal{O}_0\},\ d\mathcal{O}_0 = \{\mathcal{Q}, \mathcal{O}_1\},\ d\mathcal{O}_1 = \{\mathcal{Q}, \mathcal{O}_2\}, \\ d\mathcal{O}_2 = \{\mathcal{Q}, \mathcal{O}_3\},\ d\mathcal{O}_3 = \{\mathcal{Q}, \mathcal{O}_4\},\ d\mathcal{O}_4 = 0\ . \end{aligned} \tag{5.1.7}$$

In our case, we consider a $U(1)^2$-equivariant version of the topological descent procedure, that is to choose $\mathcal{Q}$ so that $\mathcal{Q}^2 = \mathcal{L}_V$ and also change $d \to D \equiv d + \iota_V$ to obtain the operator associated to the two-cycle. In terms of the component fields, it can be written as [103]

$$\mathcal{O}_{\mathbb{P}^1} = \int_{\mathbb{P}^1} \mathcal{O}_2 = \int_{M_4} \left\{ \omega \wedge \mathrm{Tr}\Big(\Phi F + \frac{1}{2}\psi \wedge \psi\Big) + H\, \mathrm{Tr}\Big(F \wedge F\Big) \right\}\ . \tag{5.1.8}$$

Here $\omega$ and $H$ are the Kähler two-form on the $\mathbb{P}^1$ and the moment map $\iota_V \omega = dH$, respectively. $M_4$ denotes the spacetime. The first part of (5.1.8) without $H$ is the non-equivariant version of the topological operator associated to two-cycle. It is convenient to study the generating function $\langle e^{t\,\mathcal{O}_{\mathbb{P}^1}} \rangle$ of the correlators $\langle \mathcal{O}_{\mathbb{P}^1} \ldots \mathcal{O}_{\mathbb{P}^1} \rangle$. This causes a shift of the instanton parameter by $q \to q \exp(tH)$ at the fixed points of the blow-up $\hat{\mathbb{C}}^2$ [77–79]. The expectation value of the generating function can be written as

$$\hat{\mathcal{Z}}^t \equiv \langle e^{t\mathcal{O}_{\mathbb{P}^1}} \rangle = \sum_{\vec{k} \in \Lambda} \mathcal{Z}^{(N),t}(\vec{k}) \cdot \mathcal{Z}^{(S),t}(\vec{k})\ , \tag{5.1.9}$$

where

$$\begin{aligned} \mathcal{Z}^{(N),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1, q \exp(t\epsilon_1), \vec{m} - \tfrac{1}{2}\epsilon_1)\ , \\ \mathcal{Z}^{(S),t}(\vec{k}) &\equiv \mathcal{Z}(\vec{a} + \vec{k}\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2, q \exp(t\epsilon_2), \vec{m} - \tfrac{1}{2}\epsilon_2)\ . \end{aligned} \tag{5.1.10}$$

Now, as we shrink the two-cycle $\mathbb{P}^1$ to recover the flat $\mathbb{C}^2$, the effect of inserting $(\mathcal{O}_{\mathbb{P}^1})^d$ turns out to give a vanishing contribution for small $d$ due to the selection rule. We recall that the instanton breaks the $U(1)_R$ symmetry to the discrete subgroup $\mathbb{Z}_{2b_0}$ with $b_0 = 2h^\vee - \sum_l I_2(\mathbf{R}_l)$ where the sum is over all hypermultiplets, and $h^\vee$ is the dual Coxeter number of the gauge group and $\mathbf{R}_l$ denotes the representation of the $l$-th hypermultiplet and $I_2(\mathbf{R})$ being the quadratic Dynkin index.[3] The first term of the operator $\mathcal{O}_{\mathbb{P}^1}$ (the two-form piece) carries $R$-charge $+2$, which is the familiar non-equivariant version. This discrete $R$-charge is sometimes called as a ghost number. The correlation functions vanish unless the $R$-charges add up to zero, modulo $2b_0 = 4h^\vee - 2\sum_l I_2(\mathbf{R}_l)$. Therefore, expanding (5.1.9) in powers of $t$, we find

$$\boxed{\langle e^{t\mathcal{O}_{\mathbb{P}^1}} \rangle = \mathcal{Z} + \mathcal{O}\left(t^{2h^\vee - \sum_l I_2(\mathbf{R}_l)}\right)} \,. \tag{5.1.11}$$

This is our blowup equation. To show this, notice that each term at order $t^m$ carries pieces with $R$-charge between 0 and $2m$. When $m < b_0$, the only possible non-trivial contribution comes from the $R = 0$ piece $\int HF \wedge F$ at zero instanton sector. This piece vanishes for zero instanton sector (at the north/south poles). For $n$-instanton sector, one should have $R = 2b_0 n$, which is the condition to absorb the fermionic zero modes. For $m \geq b_0$, we always have a term that absorbs all the fermionic zero modes (or the term that has $R \equiv 0 \bmod 2b_0 n$) so they do not vanish.

We see that as long as the hypermultiplet representation is not too large, *i.e.*, when $b_0 = 2h^\vee - \sum_l I_2(\mathbf{R}_l) > 2$, this allows us to write 3 independent relations for the 3 unknown variables. One can expand $\langle e^{t\,\mathcal{O}_{\mathbb{P}_1}} \rangle$ to order $t^2$, $\mathcal{O}(t^2)$ and then recursively solve for $\mathcal{Z}$ at each instanton number. So the instanton part of the partition function will be completely determined from the perturbative partition function. An explicit form of the recursion relation will be studied in Section 5.1.2.

[3] We normalize it so that $I_2(\mathbf{F}) = 1$ for the fundamental representation $\mathbf{F}$.

**Correlation functions in 5d** We now turn to 5d $\mathcal{N}=1$ gauge theory wrapped on $S^1$. The Casimir invariant $\mathrm{Tr}(\Phi^2)$ and its descendants are no longer considered as well-defined observables. Instead, there are two types of $\mathcal{Q}$-invariant observables [104]. The first type of observables are constructed from the 5d Wilson loop on the $S^1$ by applying the descent procedure. The second type of observables introduce the 3d (Kähler) Chern-Simons term, which can be written as [104, 115]

$$\mathcal{O}_{\mathbb{P}^1} = \exp\Bigg[\int_{S^1\times M_4}\Big(\omega\wedge\mathrm{Tr}\Big(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\Big) \\ +\omega\wedge\Big(\phi F+\frac{1}{2}\psi\wedge\psi\Big)\wedge dt+H\,\mathrm{Tr}\Big(F\wedge F\Big)\wedge dt\Big)\Bigg]\ . \tag{5.1.12}$$

It can be viewed as the natural $S^1$ uplift of (5.1.8) via exponentiation. The correlation function is now given by

$$\hat{\mathcal{Z}}^d \equiv \langle(\mathcal{O}_{\mathbb{P}^1})^d\rangle = \sum_{\vec{k}\in\Lambda}\mathcal{Z}^{(N),d}(\vec{k})\cdot\mathcal{Z}^{(S),d}(\vec{k})\ , \tag{5.1.13}$$

where

$$\begin{aligned}\mathcal{Z}^{(N),d}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_1,\epsilon_1,\epsilon_2-\epsilon_1,q\exp\left((d-\tfrac{b}{2})\epsilon_1\right),\vec{m}-\tfrac{1}{2}\epsilon_1)\ ,\\ \mathcal{Z}^{(S),d}(\vec{k}) &\equiv \mathcal{Z}(\vec{a}+\vec{k}\epsilon_2,\epsilon_1-\epsilon_2,\epsilon_2,q\exp\left((d-\tfrac{b}{2})\epsilon_2\right),\vec{m}-\tfrac{1}{2}\epsilon_2)\ .\end{aligned} \tag{5.1.14}$$

Here the quantity $b$ is given as

$$b\equiv h^\vee-\frac{1}{2}\sum_i I_2(\mathbf{R}_i)-\kappa_{\text{eff}}\ ,\quad \kappa_{\text{eff}}=\kappa-\frac{1}{2}\sum_i I_3(\mathbf{R}_i)\ , \tag{5.1.15}$$

where $I_2(\mathbf{R})$ and $I_3(\mathbf{R})$ are quadratic and cubic Casimir invariants respectively. We note that $d$ appearing in the exponential in (5.1.14) has to be an integer to be gauge-invariant.

The reason that the instanton parameter is further shifted by $\exp(\frac{b}{2}H|_p)$ is that the instanton mass parameter is twisted by $SU(2)_R$ as in the case of the hypermultiplet mass. The $SU(2)_R$ twisted mass of the instanton soliton is given by $m_{\text{inst}} \equiv m_{0,\text{eff}} - \kappa_{\text{eff}}\,\epsilon_+$. The effective Chern-Simons coupling $\kappa_{\text{eff}}$ also induces an electric charge to the instanton, contributing to its ground state energy as $E_0 = m_{\text{inst}} - \vec{a}\cdot\vec{\Pi}$, where $\vec{\Pi}$ is the $U(1)^r \subset G$ electric charge.[4] To keep the effective instanton mass $m_{\text{inst}}$ invariant at a fixed point $p$ of the blow-up $\hat{\mathbb{C}}^2$, we require the shifted gauge coupling $m_0|_p$ to be

$$m_0|_p = m_0 + \frac{b}{2}H|_p \quad \text{with} \quad b \equiv h^\vee - \sum_i \frac{I_2(\mathbf{R}_i)}{2} - \kappa_{\text{eff}}. \tag{5.1.16}$$

For the case of 5d pure $\mathcal{N}=1$ SYM, the correlation function turns out to be

$$\boxed{\langle (\mathcal{O}_{\mathbb{P}^1})^d \rangle = \mathcal{Z} \quad \text{for} \quad 0 \le d \le d_{\max}}\,, \tag{5.1.17}$$

where $d_{\max} = h^\vee$.[5] We call (5.1.17) as the blowup equation. The value of $d_{\max}$ depends on the matter content and gauge group. For $d_{\max} \ge 2$, there are a sufficient number of algebraic relations to determine the instanton partition function recursively in increasing order of instantons. This fact was utilized in [90] to compute instanton partition function for the gauge theories with exceptional gauge groups, for which the ADHM construction of instanton moduli space is unknown.

In this paper, we aim at developing the relation (5.1.17) for various 5d $\mathcal{N}=1$ gauge theories with hypermultiplets in various representations, so as to compute the instanton partition function. We will identify a certain bound on $d$ in Section 5.1.3 as the *necessary* condition for (5.1.17) for a large number of theories. We conjecture that the bound on $d$ we obtain is actually sufficient to obtain the blowup equation (5.1.17). While we do not attempt to prove

[4] This agrees with the supersymmetric Casimir energy of the ADHM quantum mechanics.
[5] This was shown in [79] for the case of $G = SU(N)$.

this sufficiency, we compute $n$-instanton partition function $Z_n$, based on the recursion formula that will be derived shortly from (5.1.17), and confirm the agreement with the known result obtained from an alternative method.

We find a universal expression for the bound on $d$ when the gauge group is neither $SU(N)$ nor $Sp(N)$:

$$d_{\max} = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) \quad \text{for } G \neq SU(N) \text{ or } Sp(N). \tag{5.1.18}$$

This is essentially identical condition as in $4d$ $\mathcal{N}=2$ gauge theory. But in 5d, some new effects come into play. For the $SU(N)$ case, we can have a Chern-Simons term generated at 1-loop, which alters the bound on $d$. When there is neither bare nor effective Chern-Simons coupling, the same bound holds for the $SU(N)$ case as well. The detailed condition will be given in section 5.1.3. For the case of $Sp(N)$, one can turn on the discrete $\theta$-parameter and it turns out the bound on $d$ depends on this parameter.

### 5.1.2 Recursion formula for 5d instanton partition function

The blowup equation (5.1.17) can be translated to a recursion formula on the (5d) $n$-instanton contribution $Z_n$ to the full partition function $\mathcal{Z}$. To derive this, we decompose the partition function $\mathcal{Z}$ in terms of the classical, one-loop, and instanton pieces:

$$\mathcal{Z}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m}) = Z_{\text{class}}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m})\cdot Z_{\text{1-loop}}(\vec{a},\epsilon_1,\epsilon_2,\vec{m})\cdot Z_{\text{inst}}(\vec{a},\epsilon_1,\epsilon_2,q,\vec{m}), \tag{5.1.19}$$

where $Z_{\text{inst}}$ can be further expanded in terms of the instanton fugacity $q$ as[6]

$$Z_{\text{inst}}(\vec{a}, \epsilon_1, \epsilon_2, q, \vec{m}) = \sum_{n\geq 0} q^n Z_n(\vec{a}, \epsilon_1, \epsilon_2, m) \ . \tag{5.1.20}$$

Then the blowup equation (5.1.17) can be written as

$$\begin{aligned} Z_{\text{inst}} &= \sum_{\vec{k}} \left[ \frac{Z^{(N),d}_{\text{class}}(\vec{k})\, Z^{(S),d}_{\text{class}}(\vec{k})}{Z_{\text{class}}} \frac{Z^{(N),d}_{\text{1-loop}}(\vec{k})\, Z^{(S),d}_{\text{1-loop}}(\vec{k})}{Z_{\text{1-loop}}} \right] Z^{(N),d}_{\text{inst}}(\vec{k})\, Z^{(S),d}_{\text{inst}}(\vec{k}) \\ &\equiv \sum_{\vec{k}} f_d(\vec{k}) Z^{(N),d}_{\text{inst}}(\vec{k})\, Z^{(S),d}_{\text{inst}}(\vec{k}) \ , \end{aligned} \tag{5.1.21}$$

where the superscript $(N/S), d$ denotes the appropriate shift of the parameters, specified in (5.1.10). The function $f_d(\vec{k})$ is determined only via the perturbative part of the partition function.

We recall the known expressions for the classical and 1-loop partition func-

[6] Sometimes the instanton partition function is expanded in powers of the shifted instanton mass $q\exp(-b\frac{\epsilon_1+\epsilon_2}{2})$ instead of $q$ [79, 81, 90]. We expand it with the true instanton fugacity, which makes the symmetry property $\epsilon_{1,2} \to -\epsilon_{1,2}$ of $Z_n$ manifest. This is the one that we obtain using the ADHM quantum mechanics.

tion (in 5d) [3, 24, 32]:[7]

$$Z_{\text{class}} = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\left(\frac{1}{2}m_0\, h_{ij}a_i a_j + \frac{\kappa}{6}d_{ijk}a^i a^j a^k\right)\right], \tag{5.1.22}$$

$$\begin{aligned} Z^{\text{vec}}_{\text{1-loop}} &= \exp\left[\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{\alpha}\in\Delta}\Big(\frac{(\vec{a}\cdot\vec{\alpha}+\epsilon_+)^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\alpha}+\epsilon_+)+1\Big)\right] \\ &\quad\times \text{PE}\left[-\frac{p_1p_2}{(1-p_1)(1-p_2)}\sum_{\vec{\alpha}\in\Delta}e^{-\vec{a}\cdot\vec{\alpha}}\right] \qquad \text{for the vector multiplet} \end{aligned} \tag{5.1.23}$$

$$\begin{aligned} Z^{\text{hyp},l}_{\text{1-loop}} &= \exp\left[-\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{\omega}\in\mathbf{R}_l}\Big(\frac{(\vec{a}\cdot\vec{\omega}+m_l)^3}{12} - \frac{\epsilon_1^2+\epsilon_2^2+24}{48}(\vec{a}\cdot\vec{\omega}+m_l)+1\Big)\right] \\ &\quad\times \text{PE}\left[+\frac{(p_1p_2)^{\frac{1}{2}}\cdot y_\ell}{(1-p_1)(1-p_2)}\sum_{\vec{\omega}\in\mathbf{R}_l}e^{-\vec{a}\cdot\vec{\omega}}\right] \qquad \text{for the } l\text{'th hypermultiplet} \end{aligned} \tag{5.1.24}$$

where $p_1 \equiv e^{-\epsilon_1}$, $p_2 \equiv e^{-\epsilon_2}$, $y_l \equiv e^{-m_l}$, $q \equiv e^{-m_0}$.[8] Also $\Delta$ is the set of all roots and $\vec{\omega}$ runs over all weight vectors in representation $\mathbf{R}_\ell$. Here, PE represents the Plethystic exponential

$$\text{PE}\left[f(\vec{a},\epsilon_1,\epsilon_2,m_0,\vec{m})\right] \equiv \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f(n\vec{a},n\epsilon_1,n\epsilon_2,nm_0,n\vec{m})\right). \tag{5.1.25}$$

We also set the radius of $S^1$ as $\beta = 1$. Also, the symbols $h_{ij}$ and $d_{ijk}$ are defined as

$$h_{ij} = \text{Tr}(T_iT_j)\ , \quad d_{ijk} = \frac{1}{2}\text{Tr}T_i\{T_j,T_k\}\ , \tag{5.1.26}$$

[7]There exists an ambiguity in writing the perturbative partition function, which depends on a choice of the $\mathbb{C}^2$ boundary condition at infinity. The equations (5.1.23) and (5.1.24) are fixed upon a specific choice. The 'Casimir part' of $Z_{\text{1-loop}}$ is included here to make $f_d(\vec{k})_{\text{1-loop}}$ and thus the whole blow-up equations respect the charge conjugation, regardless of the ambiguity. We thank Hee-Cheol Kim for the related comment.

[8]We assume a particular Weyl chamber in the Coulomb branch, i.e., $0 < a_i < \epsilon_+ < m$ for all $i \in \{1,\cdots,r\}$.

where $T_i$ are the generators of the gauge algebra. They satisfy the relations

$$\begin{aligned}\sum_{\vec{\omega}\in\mathbf{R}}(\vec{a}\cdot\vec{\omega})(\vec{b}\cdot\vec{\omega})(\vec{c}\cdot\vec{\omega}) &= I_3(\mathbf{R})\, d_{ijk}\, a^i b^j c^k,\\ \sum_{\vec{\omega}\in\mathbf{R}}(\vec{a}\cdot\vec{\omega})(\vec{b}\cdot\vec{\omega}) &= I_2(\mathbf{R})\, h_{ij}\, a^i b^j,\\ \sum_{\vec{\omega}\in\mathbf{R}}(\vec{a}\cdot\vec{\omega}) &= 0,\end{aligned} \tag{5.1.27}$$

where $I_2(\mathbf{R})$ and $I_3(\mathbf{R})$ are the quadratic and cubic Dynkin indices.

Substituting them to (5.1.21), we obtain the ratio of three different $Z$'s given as

$$f_d(\vec{k})_{\text{class}} = q^{\frac{\vec{k}\cdot\vec{k}}{2}}\,(p_1p_2)^{(\frac{b}{2}-d)(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa}{6}d_{ijk}\,k^ik^jk^k}\times e^{-(\frac{b}{2}-d)(\vec{a}\cdot\vec{k})}\,e^{-\frac{\kappa}{2}d_{ijk}\,a^ik^jk^k}\ , \tag{5.1.28}$$

$$f_d(\vec{k})^{\text{vec}}_{\text{1-loop}} = e^{\frac{h^\vee}{2}(\vec{a}\cdot\vec{k})}\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)^{-1}\ , \tag{5.1.29}$$

$$\begin{aligned}f_d(\vec{k})^{\text{hyp}}_{\text{1-loop}} = {}& e^{-\frac{I_2(\mathbf{R}_l)}{4}(\vec{a}\cdot\vec{k})+\frac{I_3(\mathbf{R}_l)}{4}d_{ijk}\,a^ik^jk^k}\,(p_1p_2)^{\frac{I_2(\mathbf{R}_\ell)}{8}(\vec{k}\cdot\vec{k})-\frac{I_3(\mathbf{R}_l)}{12}d_{ijk}\,k^ik^jk^k}\\ &\times y_\ell^{-\frac{I_2(\mathbf{R}_\ell)}{4}(\vec{k}\cdot\vec{k})}\prod_{\omega\in\mathbf{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\text{tw},l},\epsilon_1,\epsilon_2)\ ,\end{aligned} \tag{5.1.30}$$

where we split the $f_d(\vec{k})$ into classical and 1-loop pieces for vector and hypermultiplet. Here we used $I_2(\mathbf{adj}) = 2h^\vee$, $I_3(\mathbf{adj}) = 0$, and also the fact $h_{ij}$ and $d_{ijk}$ are totally symmetric. We also define $m_{\text{tw}} \equiv m - \epsilon_+$. The function $\mathcal{L}_k(x,\epsilon_1,\epsilon_2)$ is introduced to denote concisely the combination of the PE parts:

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) \equiv \text{PE}\left[e^{-x}\left(\frac{p_1^k\,p_2}{(1-p_1)(1-\frac{p_2}{p_1})}+\frac{p_1\,p_2^k}{(1-\frac{p_1}{p_2})(1-p_2)}-\frac{p_1p_2}{(1-p_1)(1-p_2)}\right)\right]. \tag{5.1.31}$$

One can easily check that the expression inside the PE vanishes at $k = 0, 1$.

After some work, it is not difficult to find that

$$\mathcal{L}_k(x,\epsilon_1,\epsilon_2) = \begin{cases} \prod_{m+n\le k-2} (1-p_1^{m+1}p_2^{n+1}e^{-x}) & \text{for } k \ge +2 \\ \prod_{m+n\le -k-1} (1-p_1^{-m}p_2^{-n}e^{-x}) & \text{for } k \le -1 \\ 1 & \text{for } k=0,1. \end{cases} \quad . \tag{5.1.32}$$

Combining them all together, the recursion formula on the $n$-instanton piece $Z_n$ can be written as

$$Z_n = \sum_{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n} \Bigg( (p_1p_2)^{(\frac{b}{2}-d)(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa_{\rm eff}}{6}d_{ijk}\,k^ik^jk^k} e^{(d+\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\rm eff}}{2}d_{ijk}\,a^ik^jk^k}$$
$$\times \frac{\prod_l y_{{\rm tw},l}^{-I_2(\mathbf{R}_l)(\frac{\vec{k}\cdot\vec{k}}{4})}\prod_{\omega\in\mathbf{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{{\rm tw},l},\epsilon_1,\epsilon_2)}{\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)} \cdot p_1^{(\frac{b}{2}-d)\ell}p_2^{(\frac{b}{2}-d)m} Z_\ell^{(N)}(\vec{k})Z_m^{(S)}(\vec{k}) \Bigg), \tag{5.1.33}$$

where $y_{{\rm tw},l} \equiv e^{-m_{{\rm tw},l}} = y_l/\sqrt{p_1p_2}$ and $l$ runs over all hypermultiplets in the theory. This is a generalization of the recursion formula found for the pure SYM case [79, 81].

**Solving the recursion formulae** The recursion relation (5.1.33) can be rewritten as

$$Z_n = p_1^{n(\frac{b}{2}-d)}Z_n^{(N)} + p_2^{n(\frac{b}{2}-d)}Z_n^{(S)} + I_n^{(d)} \qquad \text{with an allowed range of } d, \tag{5.1.34}$$

where $I_n^{(d)}$ is defined as

$$I_n^{(d)} = \sum_{\substack{\frac{1}{2}\vec{k}\cdot\vec{k}+\ell+m=n \\ \ell,m\neq n}} \Bigg( (p_1p_2)^{(\frac{b}{2}-d)(\frac{\vec{k}\cdot\vec{k}}{2})+\frac{\kappa_{\rm eff}}{6}d_{ijk}\,k^ik^jk^k} e^{(d+\frac{\kappa_{\rm eff}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\rm eff}}{2}d_{ijk}\,a^ik^jk^k}$$
$$\times \frac{\prod_l y_{{\rm tw},l}^{-I_2(\mathbf{R}_l)(\vec{k}\cdot\vec{k}/4)}\prod_{\omega\in\mathbf{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{{\rm tw},l},\epsilon_1,\epsilon_2)}{\prod_{\alpha\in\Delta}\mathcal{L}_{\vec{k}\cdot\vec{\alpha}}(\vec{a}\cdot\vec{\alpha},\epsilon_1,\epsilon_2)} \cdot p_1^{(\frac{b}{2}-d)\ell}p_2^{(\frac{b}{2}-d)m} Z_\ell^{(N)}(\vec{k})Z_m^{(S)}(\vec{k}) \Bigg). \tag{5.1.35}$$

Notice that we have a set of equations labeled by the parameter $d$. If the blowup equation holds for at least 3 values of $d$, we can solve it for $Z_n$. The $n$-instanton partition function $Z_n$ is given as the solution to the three linear equations (5.1.34) with consecutive integers $\{d_0, d_0+1, d_0+2\}$,

$$Z_n(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) = \frac{p_1^n p_2^n\, I_n^{(d_0+2)} - (p_1^n + p_2^n)\, I_n^{(d_0+1)} + I_n^{(d_0)}}{(1-p_1^n)(1-p_2^n)}. \tag{5.1.36}$$

Since $I_n^{(d)}$ only involves low-order instanton corrections, the $n$-instanton partition function $Z_n$ can be constructed from $Z_{m<n}$, allowing us to obtain the full non-perturbative part $Z_{\text{inst}}$ in a recursive manner starting from $Z_0 = 1$.

Therefore we arrive at a remarkable conclusion. The *non-perturbative* partition function $Z_{\text{inst}}$ is completely fixed by the *perturbative* partition function! We note that we do not reach this conclusion by requiring the perturbative series to be well-behaved, as is often done in the resurgence analysis. Instead, we demand consistency upon smooth deformation of the spacetime $\mathbb{C}^2$ or $\mathbb{C}^2 \times S^1$. Such consistency condition requires non-perturbative parts to exist and even enough to fix the instanton partition function (at least for a large number of examples).

Now, let us write the solution for 1-instanton explicitly. At one instanton level, the formula (5.1.35) can be written as

$$I_1^{(d)} = \sum_{\vec{k}\in\Delta_\ell} \Bigg( (p_1p_2)^{(\frac{b}{2}-d)} e^{(d+\frac{\kappa_{\text{eff}}}{2})(\vec{a}\cdot\vec{k})}\, e^{-\frac{\kappa_{\text{eff}}}{2} d_{ijk}\, a^i k^j k^k} \times \frac{\prod_l y_{\text{tw},l}^{-I_2(\mathbf{R}_l)/2} \prod_{\omega\in\mathbf{R}_l} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega} + m_{\text{tw},l}, \epsilon_1, \epsilon_2)}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-p_1^{-1}e^{\vec{a}\cdot\vec{k}})(1-p_2^{-1}e^{\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}}) \prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})} \Bigg), \tag{5.1.37}$$

where $\Delta_\ell$ is the set of long roots ($\vec{k}\cdot\vec{k} = 2$) and we used $Z_0 = 1$. It turns out to be more convenient to express $Z_1$ by decomposing $I_1^{(d)}$ into the flux sum, *i.e.*,

$I_1^{(d)} \equiv \sum_{\vec{k}\in\Delta_\ell} i_1^{(d)}(\vec{k})$, where

$$i_1^{(d)}(\vec{k}) \equiv (p_1p_2)^{(\frac{b}{2}-d)}\, e^{(d+\frac{\kappa_{\text{eff}}}{2})(\vec{a}\cdot\vec{k})} e^{-\frac{\kappa_{\text{eff}}}{2}d_{ijk}a^ik^jk^k} \\ \times \frac{\prod_l (y_l^{\text{tw}})^{-I_2(\mathbf{R}_l)/2}\prod_{\omega\in\mathbf{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\text{tw},l},\epsilon_1,\epsilon_2)}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})(1-p_1^{-1}e^{\vec{a}\cdot\vec{k}})(1-p_2^{-1}e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})}. \tag{5.1.38}$$

Using the property $i_1^{(d_0+\aleph)}(\vec{k})/i_1^{(d_0)}(\vec{k}) = (p_1p_2)^{-\aleph}\, e^{\aleph(\vec{a}\cdot\vec{k})}$, the one-instanton partition function $Z_1$ can be written as

$$\begin{aligned} Z_1 &= \sum_{\vec{k}\in\Delta_\ell} \frac{(1-p_1^{-1}e^{\vec{a}\cdot\vec{k}})(1-p_2^{-1}e^{\vec{a}\cdot\vec{k}})}{(1-p_1)(1-p_2)} \cdot i_1^{(d_0)}(\vec{k}) \qquad (5.1.39)\\ &= \frac{(p_1p_2)^{(\frac{b}{2}-d_0)}\prod_l (y_l^{\text{tw}})^{-\frac{I_2(\mathbf{R}_l)}{2}}}{(1-p_1)(1-p_2)} \sum_{\vec{k}\in\Delta_l} \frac{e^{(d_0+\frac{\kappa_{\text{eff}}}{2})(\vec{a}\cdot\vec{k})-\frac{\kappa_{\text{eff}}}{2}d_{ijk}a^ik^jk^k}\prod_{\omega\in\mathbf{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_l^{\text{tw}})}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})}. \end{aligned}$$

Notice that there are multiple options for choosing $d_0$. However, we find that (5.1.39) is independent of a specific choice of $d_0$. Once we choose $d_0 = 0$, for instance, which works in most cases,[9] (5.1.39) becomes

$$\boxed{Z_1 = \frac{(p_1p_2)^{\frac{b}{2}}\prod_l (y_l^{\text{tw}})^{-\frac{I_2(\mathbf{R}_l)}{2}}}{(1-p_1)(1-p_2)} \sum_{\vec{k}\in\Delta_\ell} \frac{e^{\frac{\kappa_{\text{eff}}}{2}(\vec{a}\cdot\vec{k}-d_{ijk}a^ik^jk^k)}\prod_{\omega\in\mathbf{R}_l}\mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_l^{\text{tw}})}{(1-p_1p_2e^{-\vec{a}\cdot\vec{k}})(1-e^{\vec{a}\cdot\vec{k}})\prod_{\vec{\alpha}\cdot\vec{k}=-1}(1-e^{-\vec{a}\cdot\vec{\alpha}})}}\,. \tag{5.1.40}$$

When the hypermultiplets are in the representations with $|\vec{k}\cdot\vec{w}| \leq 1$ for all $\vec{w}\in\mathbf{R}$, we have

$$\prod_{\omega\in\mathbf{R}} \mathcal{L}_{\vec{k}\cdot\vec{\omega}}(\vec{a}\cdot\vec{\omega}+m_{\text{tw}},\epsilon_1,\epsilon_2) = \prod_{\vec{k}\cdot\vec{\omega}=-1}(1-y_{\text{tw}}e^{-\vec{a}\cdot\vec{\omega}}). \tag{5.1.41}$$

The formula (5.1.39) indeed reduces to the pure YM partition function derived in [90, 100] upon removing hypermultiplets and Chern-Simons levels up to the

[9] A numerical value of $d_0$ should be a half-integer for theories with $G = Sp(N)_{\theta=\pi}$.

overall factor $(p_1p_2)^{\frac{b}{2}} = e^{-\frac{h^\vee}{2}(\epsilon_1+\epsilon_2)}$ that accounts for the shift of instanton fugacity.

We claim that (5.1.39) is the closed-form expression for the one-instanton partition function, which holds *universally for any gauge theory* with $d_{\max} > 2$. In section 5.1.3, we study the structure of the blowup equations to bound the number of possible independent equations.

### 5.1.3 Number of independent blowup equations

We are mainly interested in 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ gauge theories which are UV-complete. The UV-complete set of 4d $\mathcal{N} = 2$ gauge theories are classified in [121]. For 5d gauge theories that are UV complete as 5d SCFTs, possible matter representations are restricted to [34]:[10]

- fundamental representation for $SU(N)$, $SO(N)$, $Sp(N)$, $G_2$, $F_4$, $E_6$, $E_7$
- antisymmetric representation for $SU(N)$, $Sp(N)$
- spinor representation for $SO(N)$ with $7 \leq N \leq 14$
- rank-3 antisymmetric representation for $Sp(3)$, $Sp(4)$, $SU(6)$, $SU(7)$
- symmetric representation for $SU(N)$.

In the case of 4d, we can also have the following additional cases:

- adjoint representation for arbitrary group
- rank-3 antisymmetric for $SU(8)$
- **16** for $Sp(2)$ (half-hypermultiplet)

[10] A gauge group is always assumed to be simple in the current paper.

We note that though our blow-up formula is applicable to a large number of 5d theories with various matter representations, we are not able to apply our formula for some cases including the one with adjoint hypermultiplet since the number of independent blowup equations is smaller than 3.

The formula (5.1.33) is valid only for a certain range of $d$, for which $\langle (\mathcal{O}_{\mathbb{P}^1})^d \rangle = \mathcal{Z}$. We want to narrow down the valid range of $d$ by performing a simple sanity check on the blowup equation for the one-instanton partition function:

$$Z_1 = p_1^{\frac{b}{2}-d} Z_1^{(N)} + p_2^{\frac{b}{2}-d} Z_1^{(S)} + I_1^{(d)} \qquad \text{with an allowed range of } d\ . \tag{5.1.42}$$

Specifically, we want to examine the expansion of each term in (5.1.42) in powers of $p_1 p_2 \ll 1$. The leading exponent of each term behaves as

$$\begin{aligned} I_1^{(d)} \sim & \begin{cases} g_0(\vec{a}, \vec{m}_{\text{tw}}) \cdot (p_1 p_2)^{\frac{b}{2}-d+1} + \cdots & \text{for } N_{\text{sym}} = 0 \\ g_0(\vec{a}, \vec{m}_{\text{tw}}) \cdot (p_1 p_2)^{\frac{b}{2}-d} + \cdots & \text{for } N_{\text{sym}} = 1 \end{cases} \\ Z_1 \sim \quad & g_1(\vec{a}, \vec{m}_{\text{tw}}) \cdot (p_1 p_2)^{\frac{s}{2}} + \cdots \\ p_1^{\frac{b}{2}-d} Z_1^{(N)} \sim p_2^{\frac{b}{2}-d} Z_1^{(S)} \sim \quad & g_2(\vec{a}, \vec{m}_{\text{tw}}) \cdot (p_1 p_2)^{\frac{b}{4}-\frac{d}{2}+\frac{s}{4}} + \cdots\ , \end{aligned} \tag{5.1.43}$$

where $g_{0,1,2}(\vec{a}, \vec{m}_{\text{tw}})$ are functions independent of $p_{1,2}$ and $N_{\text{sym}}$ denotes the number of symmetric representation. The numerical value of $s$ will be obtained shortly for a variety of gauge theories for which ADHM-like construction is available. Notice that for the equation (5.1.42) to be true, some terms on the right-hand side should have the leading exponent less than or equal to that of $Z_1$. Therefore, the condition $d - \frac{b}{2} \geq -\frac{s}{2}$ is naturally imposed, setting a lower bound on $d$.

Similarly, an upper bound on $d$ can be found from an expansion of (5.1.42) with respect to $1/p_1p_2 \ll 1$.[11] Each term in (5.1.42) can be written as

$$\begin{aligned} I_1^{(d)} &\sim \begin{cases} h_0(\vec{a}, \vec{m}_{\text{tw}}) \cdot (1/p_1p_2)^{d-\frac{b}{2}+1} + \cdots & \text{for } N_{\text{sym}} = 0 \\ h_0(\vec{a}, \vec{m}_{\text{tw}}) \cdot (1/p_1p_2)^{d-\frac{b}{2}} + \cdots & \text{for } N_{\text{sym}} = 1 \end{cases} \\ Z_1 &\sim \quad h_1(\vec{a}, \vec{m}_{\text{tw}}) \cdot (1/p_1p_2)^{\frac{s'}{2}} + \cdots \\ p_1^{\frac{b}{2}-d} Z_1^{(N)} \sim p_2^{\frac{b}{2}-d} Z_1^{(S)} &\sim \quad h_2(\vec{a}, \vec{m}_{\text{tw}}) \cdot (1/p_1p_2)^{\frac{d}{2}-\frac{b}{4}+\frac{s'}{4}} + \cdots . \end{aligned} \tag{5.1.44}$$

Again, for (5.1.42) to be consistent, the leading exponent of $Z_1$ should be greater than or equal to those of the terms on the right-hand side. Such a requirement imposes an upper bound on $d$, namely $\frac{s'}{2} \geq d - \frac{b}{2}$. Combining the two inequalities, one can identify the following range

$$-\frac{s}{2} + \frac{b}{2} \leq d \leq \frac{s'}{2} + \frac{b}{2}, \tag{5.1.45}$$

as a necessary condition for (5.1.42). We explicitly checked that the $n$-instanton partition function $Z_n$ actually satisfies all the $(\frac{s+s'}{2})$ recursion relations up to a certain value of $n > 1$ for numerous examples whose $Z_n$ is already known from alternative methods. This is true even though the bound (5.1.45) itself is merely a *necessary* condition found from one-instanton analysis. Based on this empirical observation, we claim that the 5d recursion formulae (5.1.33) within the above range of $d$ is true at all instanton orders.

Another remarkable thing is that a numerical value of $(s, s')$ exhibits the very simple pattern across a broad range of theories whose gauge group is not $SU(N)_\kappa$.

$$s = s' = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) \qquad \text{for} \quad G \neq SU(N)_\kappa \text{ nor } Sp(N)$$

[11]This is equivalent to assuming a different parameter regime $0 < a_i < -\epsilon_+ < m$ for all $1 \leq i \leq r$. In general, an explicit form of the 1-loop partition function (5.1.23)–(5.1.24) can change depending on a parameter regime, thus affecting (5.1.33). However, all the above expressions remain valid under flipping a sign of $\epsilon_+$, such that we can simply study the expansion of the single terms in (5.1.42) with respect to $1/p_1p_2 \ll 1$.

$$s = s' - 2\left\{\frac{N_f}{2}\right\} = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) \qquad \text{for} \quad G = Sp(N)_{\theta=0} \tag{5.1.46}$$
$$s = s' + 2\left\{\frac{N_f}{2}\right\} = h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + 1 \quad \text{for} \quad G = Sp(N)_{\theta=\pi}$$

where $\{x\} \equiv x - \lfloor x \rfloor$ denote the non-integer part of $x$. As the above numerical pattern (5.1.46) emerges for all $G \neq SU(N)_\kappa$ examples that we studied, we conjecture that (5.1.46) is generally true, thereby taking the recursion formulae (5.1.33) with

$$\begin{aligned}
&0 \leq d \leq h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) && \text{for} \quad G \neq SU(N)_\kappa \text{ nor } Sp(N),\\
&0 \leq d \leq h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + \left\{\frac{N_f}{2}\right\} && \text{for} \quad G = Sp(N)_{\theta=0},\\
&-\frac{1}{2} \leq d \leq h^\vee - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + \frac{1}{2} - \left\{\frac{N_f}{2}\right\} && \text{for} \quad G = Sp(N)_{\theta=\pi},
\end{aligned} \tag{5.1.47}$$

as a basic assumption to obtain the partition function $\mathcal{Z}$ for any $G \neq SU(N)_\kappa$ gauge theory. It would be desirable to understand from the first principle the range (5.1.47) of $d$ for which (5.1.33) holds true.

It turns out to be more difficult to characterize a general pattern behind $(s, s')$ for $SU(N)_\kappa$ gauge theories, due to extra complication caused by the 5d Chern-Simons level $\kappa$. Here we consider two particular classes of $SU(N)_\kappa$ gauge theories for illustration. For $SU(N)_\kappa + N_f\mathbf{F}$ gauge theory ($N_f$ fundamental hypermultiplets) with $N_f + 2|\kappa| \leq 2N$, we find that

$$s = \begin{cases} \dfrac{N_f}{2} & \text{if } \kappa_{\text{eff}} = N - N_f,\\ N - \dfrac{1}{2}\displaystyle\sum_l I_2(\mathbf{F}) + |\kappa_{\text{eff}}| & \text{otherwise,} \end{cases} \tag{5.1.48}$$

$$s' = \begin{cases} \dfrac{N_f}{2} & \text{if } \kappa_{\text{eff}} = -N, \\ N - \dfrac{1}{2}\displaystyle\sum_l I_2(\mathbf{F}) + |\bar{\kappa}_{\text{eff}}| & \text{otherwise,} \end{cases} \tag{5.1.49}$$

where $\bar{\kappa}_{\text{eff}} \equiv \kappa + \frac{1}{2}\sum_l I_3(\mathbf{F})$. Plugging in these values to (5.1.45), we find the range of $d$ to be

$$\begin{aligned} 0 \le d \le N \qquad & \text{if } \kappa = -N + \frac{N_f}{2}, \\ 0 \le d \le N - \frac{N_f}{2} - \kappa \qquad & \text{if } \kappa \in \left(-N + \frac{N_f}{2}, -\frac{N_f}{2}\right], \\ 0 \le d \le N \qquad & \text{if } \kappa \in \left[-\frac{N_f}{2}, +\frac{N_f}{2}\right], \\ \frac{N_f}{2} - \kappa \le d \le N \qquad & \text{if } \kappa \in \left[\frac{N_f}{2}, N - \frac{N_f}{2}\right), \\ 0 \le d \le N \qquad & \text{if } \kappa = N - \frac{N_f}{2}, \end{aligned} \tag{5.1.50}$$

which always includes the range $0 \le d \le N$. Thus the recursion formula (5.1.33) holds for at least 3 values of $d$, which is enough to determine the partition function $Z_{\text{inst}}$ completely.

For the $SU(N)_\kappa$+$N_f$**F**+1**AS** theory ($N_f$ fundamentals and 1 anti-symmetric tensor) with $N_f + 2|\kappa| \le N + 4$, we find

$$\begin{aligned} s &= \min\left(N - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) - (\kappa_{\text{eff}} - 2),\ N - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + 2\left\{\frac{\kappa_{\text{eff}}}{2}\right\}\right) \\ s' &= \min\left(N - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + \Big(\kappa_{\text{eff}} + \sum_l I_3(\mathbf{R}_l) + 2\Big),\right. \\ &\qquad \left. N - \frac{1}{2}\sum_l I_2(\mathbf{R}_l) + 2\Big\{-\frac{\kappa_{\text{eff}}}{2} + \frac{1}{2}\sum_l I_3(\mathbf{R}_l)\Big\}\right) \end{aligned} \tag{5.1.51}$$

for most cases except for

$$s = \frac{N}{2} + 2 \qquad N \in 2\mathbb{Z},\ N_f = 0,\ \kappa = \frac{N}{2} + 1,$$

<table>
<tr><th>$G$</th><th>Hypermultiplets</th><th>Conditions for $d_{\max} \geq 2$</th><th>$(s, s')$</th><th>$d$</th></tr>
<tr><td>$SU(N)_\kappa$</td><td>$N_f\mathbf{F}$</td><td>Always</td><td>(5.1.48)</td><td>(5.1.50)</td></tr>
<tr><td rowspan="2">$SU(N)_\kappa$</td><td rowspan="2">$N_f\mathbf{F} + 1\mathbf{AS}$</td><td>$N_f \leq N - 1$</td><td rowspan="2">(5.1.51)</td><td rowspan="2">(5.1.45)</td></tr>
<tr><td>$N_f = N,\ \kappa \equiv N + 1\,(\mathrm{mod}\ 2)$</td></tr>
<tr><td>$Sp(N)_{\theta=0}$</td><td>$N_f\mathbf{F} + N_a\mathbf{AS}$</td><td>$N_a(N-1) + \lfloor N_f/2 \rfloor \leq N - 1$</td><td rowspan="7">(5.1.46)</td><td rowspan="7">(5.1.47)</td></tr>
<tr><td>$Sp(N)_{\theta=\pi}$</td><td>$N_f\mathbf{F} + N_a\mathbf{AS}$</td><td>$N_a(N-1) + \lceil N_f/2 \rceil \leq N$</td></tr>
<tr><td>$SO(2N)$</td><td>$N_v\mathbf{V} + N_s\mathbf{S} + N_c\mathbf{C}$</td><td>$N_v + 2^{N-4}(N_s + N_c) \leq 2N - 4$</td></tr>
<tr><td>$SO(2N+1)$</td><td>$N_v\mathbf{V} + N_s\mathbf{S}$</td><td>$N_v + 2^{N-3}N_s \leq 2N - 3$</td></tr>
<tr><td>$E_6$</td><td>$N_f\mathbf{F} + N_{\bar{f}}\bar{\mathbf{F}}$</td><td>$N_f + N_{\bar{f}} \leq 3$</td></tr>
<tr><td>$E_7$</td><td>$N_f\mathbf{F}$</td><td>$N_f \leq 2$</td></tr>
<tr><td>$E_8$</td><td>$\varnothing$</td><td></td></tr>
</table>

Table 5.1: List of 5d gauge theories whose partition function is determined via the blowup equations. The number of hypermultiplets are bounded so that there are at least 3 blowup equations. For the case of $SU(N) + N_f\mathbf{F}$ theory, it turns out that the Young diagram formula (5.2.1) always satisfy at least 3 blowup equations. When $N_f + 2|\kappa| > 2N$, however, this formula does not produce the correct partition function for the UV field theory as we discuss in the text.

$$s' = \frac{N}{2} + 2 \qquad N \in 2\mathbb{Z},\ N_f = 0,\ \kappa = -\frac{N}{2} - 1, \tag{5.1.52}$$

from which one can identify the valid range of $d$ via (5.1.45). As long as there exist at least three distinct allowed values for $d$ for given $(N, \kappa)$, the corresponding partition function $Z_{\text{inst}}$ can be solved from the recursion formula (5.1.33).

We also consider $SU(6)_\kappa + 1\mathbf{TAS}$ theory (one rank-3 antisymmetric tensor) with $|\kappa| \leq 3$ in Section 5.2. This model can be Higgsed to two disjoint copies of $SU(3)_\kappa$ theory without a bifundamental hypermultiplet [93]. At the level of the partition function, Higgsing is realized by turning off $m_{\text{tw}} = 0$ and imposing the $SU(3)$ traceless conditions. As neither of them modifies $s$ nor $s'$, the numerical value of $(s, s')$ must be identical to that of $SU(3)_\kappa$ gauge theory, the blowup equation always holds for the range $0 \leq d \leq 3$. Therefore, the recursion formula (5.1.33) is enough to determine the instanton partition function $Z_{\text{inst}}$ for $SU(6)_\kappa + 1\mathbf{TAS}$ theory as well.

We give the list of theories we consider in the current paper in Table 5.1.

## 5.2 Examples

The recursion formula (5.1.33) for the $n$-instanton partition function and also the general expression (5.1.39) at one-instanton order are widely applicable to 5d $\mathcal{N} = 1$ (and also similarly to 4d $\mathcal{N} = 2$) gauge theory whose $(s, s')$ satisfies $\frac{s+s'}{2} \geq 2$. Combined with the observation that $(s, s')$ follows (5.1.46) in most cases, they become a very efficient approach to obtaining the BPS partition function on $\mathbb{C}^2 \times S^1$ (or $\mathbb{C}^2$), unless the matter representation is 'too large.'

Conventionally, the instanton partition function can be computed by employing the ADHM construction of the instanton moduli space [1, 3, 89] or by applying the topological vertex formalism to the 5-brane web [60, 98]. Both are based on a certain UV realization of 5d $\mathcal{N} = 1$ gauge theory via geometric engineering in string theory. Even though IR 5d gauge theory sometimes can be obtained using more than one string theory realizations, the correct UV completion might be only achieved through specific string theory realizations. For instance, the $SU(2)$ gauge theory with $N_f$ fundamental hypermultiplets with $N_f \geq 5$ must be embedded into D4-D8-O8 brane system to be UV-completed as 5d $E_{N_f+1}$ Minahan-Nemeschansky SCFT [84, 122, 123]. Ordinary $(p, q)$ 5-brane web with colliding branes (without O-planes) indicate UV inconsistency [76]. A sensible QFT observable can thus be obtained only through a proper embedding of the gauge theory into string theory. In some occasions, an extra factor dressing the true QFT observable may appear during the above instanton computation, which is sensitive to the choice of a string theory embedding. Our blow-up formula (5.1.33) does not explicitly specify a particular UV completion nor string theory embedding. However, we observe that the formula does prefer a particular string theory embedding of the gauge theory. For example, for the $SU(2)$ gauge theory with $N_f$ fundamental hypermultiplets, we find the partition function obtained from the blow-up formula agrees with the partition function obtained from the ordinary $(p,q)$ 5-brane webs.

There are wide varieties of 'exceptional' gauge theories (having exceptional gauge groups or exotic matter representations) whose UV completion is found as M-theory wrapped on a singular Calabi-Yau 3-fold [114, 116–118]. As most exceptional theories lack the ADHM description [91], their instanton partition function $Z_{\text{inst}}$ has been studied in a case-by-case basis. Once the 5-brane web configuration engineering an exceptional theory is identified [45, 80, 93], the topological vertex formalism can be applied to compute the relevant partition function $\mathcal{Z}$ [98, 124]. Alternatively, one can first construct the $\mathbb{C}^2 \times T^2$ partition function for a related 6d gauge theory, based on its modularity and anomaly, then take the circle reduction to obtain the 5d partition function $\mathcal{Z}$ [18, 92]. Several interesting exceptional theories have been studied so far, based on the above two approaches. Sometimes, there exists auxiliary 4d $\mathcal{N}=2$ SCFT [125] that realizes exceptional instanton moduli space as its Higgs branch.[12] In this case, computing the superconformal index in the Higgs branch limit provides a way to compute the necessary instanton partition function for the exceptional gauge theory [64, 112, 126, 127, 130]. Likewise, 3d $\mathcal{N}=4$ theory can realize exceptional instanton moduli space via its Coulomb branch [128]. Computing its Hilbert series (or the Coulomb branch limit of the superconformal index), one can compute the instanton partition function [43, 129]. We will illustrate that bootstrapping the instanton partition function $Z_{\text{inst}}$ based on the recursion formula (5.1.33) works well for those 'exceptional' theories, providing their BPS

[12] Also 2d $\mathcal{N}=(0,4)$ version [65] for any 4d $\mathcal{N}=2$ theory can be obtained upon twisted dimensional reduction, which allows us to compute the 6d instanton string partition function.

spectrum efficiently.

### 5.2.1 Theories with known ADHM description

Let us first consider the 'standard' gauge theories with classical gauge groups, whose hypermultiplet admits UV realization as a perturbative string ending on D-branes. In these cases, the ADHM construction of the instanton moduli space is well-known [1, 3, 24]. As for the $k$-instanton partition function $Z_k$, the Witten index of the relevant ADHM quantum mechanics can be computed by SUSY localization [22, 25, 119, 120], ending up collecting all Jeffrey-Kirwan residues of a multi-dimensional contour integral. We will examine whether the recursion formula (5.1.33) actually produces the same result as the localization computation.

**SU(N)** The ADHM construction for the $n$-instanton partition function, for $SU(N)_\kappa + N_f\mathbf{F}$ ($N_f$ fundamentals) theory with $N_f + 2|\kappa| \leq 2N$ is well-known. Its partition function can be written as a sum over Young diagrams as

$$Z_n^{\text{ADHM}} = \sum_{|\vec{Y}|=n} \prod_{i=1}^{N} \prod_{\sigma \in Y_i} \frac{e^{-\kappa\phi(s)} \prod_{l=1}^{N_f} 2\sinh\frac{\phi(\sigma)+m_l}{2}}{\prod_{j=1}^{N} 2\sinh\frac{E_{ij}}{2}\, 2\sinh\frac{E_{ij}-2\epsilon_+}{2}}\ , \tag{5.2.1}$$

where

$$\begin{aligned} E_{ij}(\sigma) &= a_i - a_j - \epsilon_1 h_i(\sigma) + \epsilon_2(v_j(\sigma)+1) \\ \varphi(\sigma) &= a_i - \epsilon_+ - (n-1)\epsilon_1 - (m-1)\epsilon_2 \qquad\quad \text{for} \quad \sigma = (m,n) \in Y_i\ . \end{aligned}$$

Here $h_i(\sigma)$ denotes the distance from $\sigma$ to the right end of the diagram $Y_i$ by moving right and $v_j(\sigma)$ denotes the distance from $\sigma$ to the bottom of the diagram $Y_j$ by moving down. We checked that the instanton partition functions $Z_1$ and $Z_2$ obtained from the recursion formula (5.1.33) with (5.1.50) and the

1-instanton expression (5.1.39) precisely agree with the above $Z_{n=1,2}^{\text{ADHM}}$ for $N = 2, 3, 4$.

As we have said earlier, $Z_n^{\text{ADHM}}$ often contains an additional factor $Z_{\text{extra}}$ that captures the contribution from an extra branch of vacua of the ADHM quantum mechanics. It is sensitive to the string theory embedding (UV completion) of the gauge theory and can be regarded as spurious from the 5d QFT perspective. It is usually factorized from the true QFT partition function as

$$\sum_{n=0}^{\infty} q^n \, Z_n^{\text{ADHM}}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) = Z_{\text{QFT}}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}, q) \cdot Z_{\text{extra}}(\epsilon_1, \epsilon_2, \vec{m}, q). \quad (5.2.2)$$

A non-trivial $Z_{\text{extra}} \neq 1$ appears in the above expression (5.2.1) if and only if $N_f + 2|\kappa| = 2N$. This factor can be identified as the contribution of D1-branes escaping from D5-branes which engineer the $SU(N)_\kappa + N_f\mathbf{F}$ gauge theory. Since $Z_n = Z_n^{\text{ADHM}}$, the same factor $Z_{\text{extra}}$ emerges from the recursion formula (5.1.33) as well. The 5-brane web construction of the gauge theory is thus indirectly reflected in the recursion formula.

A similar observation is that the 1-instanton expression (5.1.39) applied to $SU(2)_\kappa + N_f\mathbf{F}$ with $N_f \geq 5$ does not match the Witten index of the D0-D4-D8-O8$^-$ quantum mechanics, which is the correct 1-instanton partition function.[13] Instead, it coincides with the topological vertex computation applied to the 5-brane web with a colliding pair of branes, which engineers the $SU(2)$ gauge theory with $N_f \geq 5$ in the IR, but behaves badly in the UV. Again, this suggests that the recursion formula (5.1.33) implicitly chooses a specific string theory construction of the gauge theory, i.e., the web of $(p, q)$ 5-branes. It would be interesting to figure out if there is a version of the recursion relation (5.1.33) that allows us to choose the particular UV embedding of the gauge theory.

[13]The case with $SU(2) \simeq Sp(1)$ is an exception, which allows $N_f \leq 7$ fundamental hypermultiplets [84].

For the $SU(N)_\kappa+N_f\mathbf{F}+1\mathbf{AS}$ theory ($N_f$ fundamental and 1 anti-symmetric hypermultiplets) with $N_f+2|\kappa|\leq N+4$, the ADHM quantum mechanics is the worldsheet theory of D1-branes, probing the D5-NS5-D7-O7$^-$ brane configuration that realizes the gauge theory. Let us compute the Witten index for 1 and 2 D1-branes, then compare with the blow-up computation based on the recursion formula (5.1.33). For instance, the Witten index for the single D1-brane can be written as

$$\begin{aligned} Z_1^{\rm ADHM} &= -\sum_{i=1}^{N}\frac{e^{-\kappa(a_i-\epsilon_+)}}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}}\frac{\prod_{l=1}^{N_f}2\sinh\frac{-\epsilon_++a_i+m_l}{2}}{2\sinh\frac{-3\epsilon_++2a_i+m_a}{2}}\prod_{j\neq i}\frac{2\sinh\frac{a_i+a_j+m_a-\epsilon_+}{2}}{2\sinh\frac{a_i-a_j}{2}\,2\sinh\frac{2\epsilon_+-a_i+a_j}{2}} \\ &-\frac{1}{2}\,\frac{e^{-\frac{\kappa}{2}(\epsilon_+-m_a)}}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}}\left(\frac{\prod_{l=1}^{N_f}2\sinh\frac{\epsilon_++2m_l-m_a}{4}}{\prod_{i=1}^{N}2\sinh\frac{3\epsilon_+-M-2a_i}{4}}-(-1)^{\kappa+\frac{N-N_f}{2}}\frac{\prod_{l=1}^{N_f}2\cosh\frac{\epsilon_++2m_l-m_a}{4}}{\prod_{i=1}^{N}2\cosh\frac{3\epsilon_+-M-2a_i}{4}}\right). \end{aligned} \tag{5.2.3}$$

Note that $Z_n^{\rm ADHM}$ contains an extra factor $Z_{\rm extra}\neq 1$ if $N_f+2|\kappa|=N+4$, coming from the spectrum of D1-branes escaping from the D5-branes on which the gauge theory is supported. The appearance of $Z_{\rm extra}\neq 1$ is an artifact of the string theory embedding, spurious from the 5d QFT perspective. We checked that $Z_1^{\rm ADHM}$ and the 1-instanton formula (5.1.39) agree for the $SU(3)$, $SU(4)$, $SU(5)$ theories whose $(n,n')$ satisfies $\frac{n+n'}{2}\geq 2$. We confirmed $Z_2=Z_2^{\rm ADHM}$ as well, where $Z_2$ is the solution of the recursion formulae (5.1.33) with (5.1.51). The same spurious factor $Z_{\rm extra}$ arises from the recursion formula, implying that our blowup equations are implicitly based on the D5-NS5-D7-O7$^-$ brane realization of the gauge theory.[14]

**Sp(N)** The $n$-instanton partition function for $Sp(N)_\theta+N_f\mathbf{F}$ theory ($\theta$ being the discrete theta-angle for $Sp$ and $N_f$ fundamental hypermultiplets) with $N_f\leq$

[14] An exceptional case is the $SU(2)$ gauge theory, in which the antisymmetric hypermultiplet decouples and never affects the recursion formula. The corresponding $Z_n$ is the same as the Young diagram formula (5.2.1).

$2N+4$ can be computed from the ADHM quantum mechanics of D1-D5-NS5-O5 branes, which engineers the gauge theory and its instantons. The Witten index for the D1-brane theory is written as

$$Z_1^{\mathrm{ADHM}} = \frac{1}{2}\frac{1}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}}\left(\frac{\prod_{l=1}^{N_f}2\sinh\frac{m_l}{2}}{\prod_{i=1}^{N}2\sinh\frac{\epsilon_+\pm a_i}{2}} + e^{i\theta}\frac{\prod_{l=1}^{N_f}2\cosh\frac{m_l}{2}}{\prod_{i=1}^{N}2\cosh\frac{\epsilon_+\pm a_i}{2}}\right). \tag{5.2.4}$$

We checked that our 1-instanton formula (5.1.39) agrees with $Z_1^{\mathrm{ADHM}}$ for the $Sp(2)$ and $Sp(3)$ gauge theories satisfying $N-1 \geq \lfloor\frac{N_f}{2}\rfloor$ (at $\theta=0$) and $N \geq \lceil\frac{N_f}{2}\rceil$ (at $\theta=\pi$). We also confirmed that $Z_2^{\mathrm{ADHM}} = Z_2$, where $Z_2$ is the solution of the recursion formulae (5.1.33) with (5.1.47). Note that there is no spurious factor $Z_{\mathrm{extra}}$ so that the ADHM and the blowup results agree $Z_n^{\mathrm{ADHM}} = Z_n$ for these theories.

For the $Sp(N)_\theta+N_f\mathbf{F}+1\mathbf{AS}$ theory ($N_f$ fundamental and 1 anti-symmetric hypermultiplets) with $N_f \leq 7$, the relevant ADHM quantum mechanics is the worldvolume gauge theory of D0-branes which probe the D4-D8-O8 brane configuration. It is well-known that the QFT on D4-branes exhibits an enhanced $E_{N_f+1}$ flavor symmetry at the UV fixed point [84]. Let us consider the Witten index for one and two D0-branes [22, 49]. For a single D0-brane, we obtain the one instanton partition function to be

$$\begin{aligned} Z_1^{\mathrm{ADHM}} &= \frac{1}{2}\frac{1}{2\sinh\frac{\epsilon_1}{2}\,2\sinh\frac{\epsilon_2}{2}\,2\sinh\frac{m_a+\epsilon_+}{2}\,2\sinh\frac{m_a-\epsilon_+}{2}} \\ &\quad\times\left(\frac{\prod_{i=1}^{N}2\sinh\frac{m_a\pm a_i}{2}\prod_{l=1}^{N_f}2\sinh\frac{m_l}{2}}{\prod_{i=1}^{N}2\sinh\frac{\epsilon_+\pm a_i}{2}} + \frac{e^{i\theta}\prod_{i=1}^{N}2\cosh\frac{m_a\pm a_i}{2}\prod_{l=1}^{N_f}2\cosh\frac{m_l}{2}}{\prod_{i=1}^{N}2\cosh\frac{\epsilon_+\pm a_i}{2}}\right). \end{aligned} \tag{5.2.5}$$

We find that $Z_1^{\mathrm{ADHM}}$ itself is *not* the same as the 1-instanton expression from the blowup (5.1.39) for the $Sp(2)_\theta$, $Sp(3)_\theta$ theories with $N_f \leq 1$ (at $\theta=0$) and $N_f \leq 2$ (at $\theta=\pi$). Instead, the difference between $Z_1$ and $Z_1^{\mathrm{ADHM}}$ can be identified as the BPS index of D0-branes moving away from the D4-D8-O8 brane system [22, 49]. Similarly, we confirmed that the 2-instanton correction $Z_2$ captures the same 5d QFT spectrum as in $Z_2^{\text{ADHM}}$, upon subtracting the spurious contribution of escaping D0-branes. It is interesting that our blow-up formula does *not* contain a spurious factor $Z_{\text{extra}}$.

**SO(N)** One can compute the instanton partition function of $SO(N) + N_v\mathbf{V}$ theory ($N_v$ hypermultiplets in the vector representation) with $N_v \leq N-4$ using the ADHM quantum mechanics of the D1-D5-NS5-O5 brane system. For even $N$, the Witten index for a single D1-brane can be written as

$$Z_1^{\text{ADHM}} = \sum_{i=1}^{N/2} \left( \frac{2\sinh(2\epsilon_+ - a_i)\, 2\sinh(a_i - \epsilon_+) \prod_{l=1}^{N_v} 2\sinh\frac{m_l \pm (a_i - \epsilon_+)}{2}}{2 \cdot 2\sinh\frac{\epsilon_1}{2}\, 2\sinh\frac{\epsilon_2}{2} \prod_{j\neq i} 2\sinh\frac{a_i \pm a_j}{2}\, 2\sinh\frac{2\epsilon_+ - a_i \pm a_j}{2}} + (a_i \to -a_i) \right). \tag{5.2.6}$$

For odd $N$,

$$Z_1^{\text{ADHM}} = \sum_{i=1}^{\lfloor N/2 \rfloor} \left( \frac{2\cosh\frac{2\epsilon_+ - a_i}{2}\, 2\sinh(a_i - \epsilon_+) \prod_{l=1}^{N_f} 2\sinh\frac{m_l \pm (a_i - \epsilon_+)}{2}}{2 \cdot 2\sinh\frac{\epsilon_1}{2}\, 2\sinh\frac{\epsilon_2}{2}\, 2\sinh\frac{a_i}{2} \prod_{j\neq i} 2\sinh\frac{a_i \pm a_j}{2}\, 2\sinh\frac{2\epsilon_+ - a_i \pm a_j}{2}} \right.$$
$$\left. + (a_i \to -a_i) \right). \tag{5.2.7}$$

The general 1-instanton expression (5.1.39) and the recursion formula (5.1.33) are applicable for all $N_v \leq N-4$. We explicitly verified that $Z_n^{\text{ADHM}} = Z_n$ for $n = 1, 2$ and $4 \leq N \leq 9$, where $Z_1$ is written in (5.1.39) and $Z_2$ is the solution of the recursion formula (5.1.33). We find that $Z_n^{\text{ADHM}} = Z_n$ involves a non-trivial extra factor $Z_{\text{extra}} \neq 1$ when $N_v = N-4$. This extra factor can be attributed to the D1-branes moving away from the D5-NS5-O5 brane system, where the 5d QFT lives. It implies that a specific UV realization of the gauge theory, i.e., type IIB string theory with D1-D5-NS5-O5, is implicit in our recursion formulae (5.1.33) with (5.1.47).

### 5.2.2 Theories with spinor hypermultiplets

So far, we have investigated the 'standard' gauge theories that have certain D-brane set-ups in type IIA/IIB string theory to realize themselves and also their instantons. For the theory with a sufficient number of the blowup equations, the $n$-instanton partition function $Z_n$ can be determined as the solution of the blowup equations. We have found that this formula agrees with the instanton counting result using the ADHM construction, modulo possible extra factor $Z_{\text{extra}}$ that is sensitive to the string theory embedding of the gauge theory.

We take advantage of the universality of the blowup equation. Recall that the blow-up recursion formula (5.1.33) holds for a certain range of $d$, i.e., the set of all integers between $0 \le d \le h^\vee - \frac{1}{2}\sum_l I(\mathbf{R}_l)$, when the gauge group $G$ is neither $SU(N)_\kappa$ nor $Sp(N)_\theta$. In this case, there is no extra complication due to the Chern-Simons level $\kappa$ or the theta angle $\theta$. One can solve the recursion formulae for the $n$-instanton correction $Z_n$ to the partition function, as long as $h^\vee - \frac{1}{2}\sum_l I(\mathbf{R}_l) \ge 2$, even for the exceptional gauge theories. We conjecture that $Z_n$ solved from the recursion formula would be the correct BPS data for UV-consistent 5d SCFTs, modulo an extra factor $Z_{\text{extra}}$ independent of the Coulomb VEV $\vec{a}$. This conjecture will be tested via comparison with [91–93] which compute $\mathcal{Z}$ for some exceptional cases.

In this section, we will focus on the $SO(N)$ gauge theories with spinor hypermultiplets. We have a sufficient number of recursion formulae (5.1.33) to determine the $n$-instanton partition function $Z_n$ of the $SO(N)$ gauge theory, if and only if

$$\begin{aligned} &N-4 \ge N_{\mathbf{v}} + 2^{\frac{N-7}{2}} \cdot N_{\mathbf{s}} && \text{for odd } N, \\ &N-4 \ge N_{\mathbf{v}} + 2^{\frac{N-8}{2}} \cdot (N_{\mathbf{s}} + N_{\mathbf{c}}) && \text{for even } N, \end{aligned} \tag{5.2.8}$$

where $N_{\mathbf{v}}$, $N_{\mathbf{s}}$, and $N_{\mathbf{c}}$ denote the number of hypermultiplets in the vector, spinor and conjugate spinor representations, respectively. Our 1-instanton expression (5.1.39) is also applicable to the cases satisfying (5.2.8). We compare our formula against any known results for $SO(N)$ gauge theory with a number of spinor hypermultiplets [91, 92]. We not only find perfect agreements for the case with the known results, but also obtain partition functions for the previously unknown cases as well.

**SO(7)** The $n$-instanton contribution $Z_n$ of $SO(7)+N_{\mathbf{s}}\,\mathbf{S}$ theory can be obtained from the SUSY quantum mechanics proposed in [91], which can be summarized as the following $SU(4)$ Young diagram expression:

$$
\begin{aligned}
Z_n^{\mathrm{YD}} = \sum_{|\vec{Y}|=n} \prod_{i=1}^{4} \prod_{s\in Y_i} & \frac{2\sinh\left(\phi(s)\right)\, 2\sinh\left(\phi(s)-\epsilon_+\right)\, \prod_{l=1}^{N_{\mathbf{s}}} 2\sinh\left(\frac{m_l\pm\phi(s)}{2}\right)}{\prod_{j=1}^{4} 2\sinh\frac{E_{ij}}{2}\, 2\sinh\frac{E_{ij}-2\epsilon_+}{2}\, 2\sinh\frac{\epsilon_+-\phi(s)-a_j}{2}} \\
\times \prod_{i\leq j}^{4} \prod_{\substack{s_{i,j}\in Y_{i,j}\\ s_i<s_j}} & \frac{2\sinh\frac{\phi(s_i)+\phi(s_j)}{2}\, 2\sinh\frac{\phi(s_i)+\phi(s_j)-2\epsilon_+}{2}}{2\sinh\frac{\epsilon_1-\phi(s_i)-\phi(s_j)}{2}\, 2\sinh\frac{\epsilon_2-\phi(s_i)-\phi(s_j)}{2}}.
\end{aligned}
\tag{5.2.9}
$$

We verified that $Z_1^{\mathrm{YD}}$ and the 1-instanton formula $Z_1$ in (5.1.39) agree for $N_{\mathbf{s}}\leq 3$. We further confirmed at two instanton order for $N_{\mathbf{s}}\leq 3$ that $Z_2^{\mathrm{YD}}=Z_2$, where $Z_2$ is the solution of the recursion formula (5.1.33) with (5.1.47). Such explicit comparison implies that the blow-up recursion formula (5.1.33) indeed works for the $SO(7)+N_{\mathbf{s}}\,\mathbf{S}$ theory.

The 1-instanton partition function of $SO(7)+4\mathbf{S}+1\mathbf{V}$ theory is given in (H.15) of [92]. From this expression, we can obtain the 1-instanton correction of $SO(7)+N_{\mathbf{s}}\,\mathbf{S}+N_{\mathbf{v}}\mathbf{V}$ theory with $(N_{\mathbf{s}},N_{\mathbf{v}})\leq(2,1)$ by integrating out hypermultiplets or equivalently taking some flavor chemical potentials to infinity. We confirmed that the result agrees with our general 1-instanton expression (5.1.39) up to order $(p_1p_2)^{13/2}$. Notice that our formula holds for any $N_{\mathbf{v}}+N_{\mathbf{s}}\leq 2$ and can be used to compute arbitrary high orders in instanton number.

**SO(8)** Our instanton formula should hold for $N_{\mathbf{v}} + N_{\mathbf{s}} + N_{\mathbf{c}} \leq 4$. Let us compare it with known results.

The 1-instanton result of $SO(8) + 1\mathbf{S} + 1\mathbf{C} + 1\mathbf{V}$ theory is found in (H.28) of [92]. It is expressed in terms of characters of irreducible representations $\chi^S_{\mathbf{R}}$, whose superscript $S \in \{G, v, s, c\}$ means either the gauge symmetry ($G$) or the flavor symmetry acting on the vector ($v$), spinor ($s$), or conjugate spinor ($c$) hypermultiplets. Their representation $\mathbf{R}$ is specified by the Dynkin label in the subscript. All irreducible characters for the flavor symmetry are assumed to be written in the orthogonal basis, to be compatible with our convention of mass parameters in (5.1.24), (5.1.33), (5.1.39). The mass parameters will be often distinguished by the superscript $S \in \{s, c, v\}$ according to the matter representation. The flavor symmetry is $Sp(N_{\mathbf{v}})_v \times Sp(N_{\mathbf{s}})_s \times Sp(N_{\mathbf{c}})_c$.

We can obtain the 1-instanton partition function of $SO(8) + N_{\mathbf{s}}\mathbf{S} + N_{\mathbf{c}}\mathbf{C} + N_{\mathbf{v}}\mathbf{V}$ theory with $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) \leq (1, 1, 1)$ from (H.28) of [92] by sending appropriate mass parameters to infinity. All the results obtained in this way is consistent with our general 1-instanton expression (5.1.39) up to $t^{20}$ order, where $t \equiv \sqrt{p_1 p_2}$. Furthermore, we are able to determine the unknown part of (H.28) of [92] as

$$\begin{aligned}
\tilde{Z}_1 = t^4 &+ \sum_{n=0}^{\infty} t^{5+2n} \chi^G_{(0n00)} \chi^v_{(1)} \chi^s_{(1)} \chi^c_{(1)} \\
&+ \sum_{n=0}^{\infty} t^{6+2n} \left( \chi^G_{(1n00)} \chi^s_{(1)} \chi^c_{(1)} + \chi^G_{(0n10)} \chi^s_{(1)} \chi^v_{(1)} + \chi^G_{(0n01)} \chi^c_{(1)} \chi^v_{(1)} \right) \\
&+ \sum_{n=0}^{\infty} t^{7+2n} \left( \chi^G_{(1n10)} \chi^s_{(1)} + \chi^G_{(1n01)} \chi^c_{(1)} + \chi^G_{(0n11)} \chi^v_{(1)} \right) - \sum_{n=0}^{\infty} t^{8+2n} \chi^G_{(1n11)} ,
\end{aligned} \tag{5.2.10}$$

where $\tilde{Z}_1 \equiv (2\sinh\frac{\epsilon_1}{2})(2\sinh\frac{\epsilon_2}{2})Z_1$ is the 1-instanton partition function with the center-of-mass factor removed.

Now we compare (5.1.39) with the 1-instanton partition function of $SO(8)+$ $2\mathbf{S}+2\mathbf{C}+2\mathbf{V}$ theory, written in (H.19) of [92]. Our 1-instanton formula (5.1.39) applied to the $SO(8)$ theories having $(N_\mathbf{s}, N_\mathbf{c}, N_\mathbf{v}) \leq (2,1,1), (1,2,2), (1,1,2),$ $(2,2,0), (2,0,2), (0,2,2)$ agree with (H.19) up to $t^{20}$ order, after suitably setting some mass parameters in (H.19) to infinity. We could further determine the unknown part of (H.19) of [92] as

$$\begin{aligned}
\tilde{Z}_1 = {} & t^{-1} - t^3(\chi^v_{(01)} + \chi^s_{(01)} + \chi^c_{(01)}) + t^5(\chi^G_{(1000)}\chi^s_{(10)}\chi^c_{(10)} + \chi^G_{(0010)}\chi^s_{(10)}\chi^v_{(10)} \\
& + \chi^G_{(0001)}\chi^c_{(10)}\chi^v_{(10)}) - t^6(\chi^G_{(1010)}\chi^s_{(10)} + \chi^G_{(1001)}\chi^c_{(10)} + \chi^G_{(0011)}\chi^v_{(10)}) + t^7\chi^G_{(1011)} \\
& - \sum_{n=0}^{\infty}\Big(t^{5+2n}\chi^G_{(0n00)}\chi^s_{(10)}\chi^c_{(10)}\chi^v_{(10)} + t^{6+2n}(\chi^G_{(1n00)}\chi^s_{(01)}\chi^c_{(01)}\chi^v_{(10)} + \chi^G_{(0n10)}\chi^s_{(01)}\chi^c_{(10)}\chi^v_{(01)} + \chi \\
& - t^{7+2n}(\chi^G_{(1n10)}\chi^s_{(01)}\chi^c_{(10)}\chi^v_{(10)} + \chi^G_{(1n01)}\chi^s_{(10)}\chi^c_{(01)}\chi^v_{(10)} + \chi^G_{(0n11)}\chi^s_{(10)}\chi^c_{(10)}\chi^v_{(01)}) \\
& + t^{8+2n}(\chi^G_{(2n10)}\chi^s_{(01)}\chi^c_{(10)} + \chi^G_{(2n01)}\chi^s_{(10)}\chi^c_{(01)} + \chi^G_{(1n20)}\chi^s_{(01)}\chi^v_{(10)} + \chi^G_{(1n02)}\chi^c_{(01)}\chi^v_{(10)} \\
& + \chi^G_{(0n21)}\chi^s_{(10)}\chi^v_{(01)} + \chi^G_{(0n12)}\chi^c_{(10)}\chi^v_{(01)}) - t^{9+2n}(\chi^G_{(2n11)}\chi^s_{(10)}\chi^c_{(10)} + \chi^G_{(1n21)}\chi^s_{(10)}\chi^v_{(10)} \\
& + \chi^G_{(1n12)}\chi^c_{(10)}\chi^v_{(10)}) + t^{10+2n}(\chi^G_{(2n21)}\chi^s_{(10)} + \chi^G_{(2n12)}\chi^c_{(10)} + \chi^G_{(1n22)}\chi^v_{(10)}) - t^{11+2n}\chi^G_{(2n22)}\Big).
\end{aligned} \tag{5.2.11}$$

Notice that (5.2.10) and (5.2.11) are manifestly invariant under the $SO(8)$ triality, transforming the $SO(8)$ representations as $(n_v n_a n_c n_s) \to (n_s n_a n_v n_c)$ along with $\chi^v_\mathbf{R} \to \chi^s_\mathbf{R} \to \chi^c_\mathbf{R} \to \chi^v_\mathbf{R}$. It can be done by shuffling the Coulomb VEVs and renaming the flavor chemical potentials. We rearranged $Z_1$ in terms of the new variables $\vec{a}'$ or $\vec{a}''$,

$$\begin{aligned}
(a'_1, a'_2, a'_3, a'_4) &= (\tfrac{-a_1+a_2+a_3-a_4}{2}, \tfrac{-a_1+a_2+a_3-a_4}{2}, \tfrac{-a_1+a_2+a_3-a_4}{2}, \tfrac{-a_1+a_2+a_3-a_4}{2}) \\
(a''_1, a''_2, a''_3, a''_4) &= (\tfrac{+a_1-a_2-a_3-a_4}{2}, \tfrac{-a_1+a_2-a_3-a_4}{2}, \tfrac{-a_1-a_2+a_3-a_4}{2}, \tfrac{+a_1+a_2+a_3-a_4}{2}),
\end{aligned} \tag{5.2.12}$$

which exchanges the $SO(8)$ irreducible characters as

$$\chi_{(n_c n_a n_s n_v)}(\vec{a}) = \chi_{(n_v n_a n_c n_s)}(\vec{a}')|_{\vec{a}'\to\vec{a}}, \quad \chi_{(n_s n_a n_v n_c)}(\vec{a}) = \chi_{(n_v n_a n_c n_s)}(\vec{a}'')|_{\vec{a}''\to\vec{a}}. \tag{5.2.13}$$

Dropping off primes from $Z_1(\vec{a}', \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v)$ or $Z_1(\vec{a}'', \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v)$, we indeed find

$$\begin{aligned} Z_1^{N_{\mathbf{s}}=N_{\mathbf{c}}=N_{\mathbf{v}}}(\vec{a}, \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v) &= Z_1^{N_{\mathbf{s}}=N_{\mathbf{c}}=N_{\mathbf{v}}}(\vec{a}', \epsilon_1, \epsilon_2; \vec{m}^v, \vec{m}^s, \vec{m}^c)|_{\vec{a}'\to\vec{a}} \\ Z_1^{N_{\mathbf{s}}=N_{\mathbf{c}}=N_{\mathbf{v}}}(\vec{a}, \epsilon_1, \epsilon_2; \vec{m}^s, \vec{m}^c, \vec{m}^v) &= Z_1^{N_{\mathbf{s}}=N_{\mathbf{c}}=N_{\mathbf{v}}}(\vec{a}'', \epsilon_1, \epsilon_2; \vec{m}^c, \vec{m}^v, \vec{m}^s)|_{\vec{a}''\to\vec{a}}, \end{aligned} \tag{5.2.14}$$

which is consistent with the triality.

Similarly, we also found the 1-instanton formula (5.1.39) applied to $SO(8)$ theories with $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) \le (4,0,0)$ or $(0,4,0)$ is compatible with the $SO(8)$ triality. Starting with the 1-instanton result $Z_1^{\text{ADHM}} = Z_1^{\text{ADHM}}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m})$ obtained from the relevant ADHM quantum mechanics for $SO(8) + N_{\mathbf{v}}\mathbf{V}$ theory with $N_{\mathbf{v}} \le 4$, we find

$$\begin{aligned} Z_1^{N_{\mathbf{c}},\, N_{\mathbf{c}}=N_{\mathbf{v}}=0}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) &= Z_1^{\text{ADHM}}(\vec{a}', \epsilon_1, \epsilon_2, \vec{m})|_{\vec{a}'\to\vec{a}} \\ Z_1^{N_{\mathbf{s}},\, N_{\mathbf{s}}=N_{\mathbf{v}}=0}(\vec{a}, \epsilon_1, \epsilon_2, \vec{m}) &= Z_1^{\text{ADHM}}(\vec{a}'', \epsilon_1, \epsilon_2, \vec{m})|_{\vec{a}''\to\vec{a}}. \end{aligned} \tag{5.2.15}$$

**SO(9)** For the $SO(9)$ theory with $N_{\mathbf{s}}$ spinor and $N_{\mathbf{v}}$ vector, our blowup formula is valid for $N_{\mathbf{v}} + 2N_{\mathbf{s}} \le 5$. The 1-instanton formula (5.1.39) can be applied to $(N_{\mathbf{s}}, N_{\mathbf{v}}) \le (1,3)$ or $(2,1)$, which has $Sp(N_{\mathbf{s}})_s \times Sp(N_{\mathbf{v}})_v$ flavor symmetry. It can be compared with the 1-instanton partition function of $SO(9) + 2\mathbf{S} + 3\mathbf{V}$ theory, which is written in (H.20) of [92] up to $t^7$ order, after appropriately taking some mass parameters to infinity. We checked all their consistency up to the given order. For example, the character expansion of $\hat{Z}_1$ for $SO(9){+}2\mathbf{S}{+}1\mathbf{V}$

can be written as

$$
\begin{aligned}
\tilde{Z}_1 =\ & t^4\chi^v_{(1)} + t^5\chi^s_{(20)} - t^6\chi^G_{(0001)}\chi^s_{(10)} \\
& + \sum_{n=0}^{\infty}\Bigg( t^{6+2n}\chi^G_{(0n00)}\chi^s_{(02)}\chi^v_{(1)} - t^{7+2n}\left(\chi^G_{(1n00)}\chi^s_{(02)} + \chi^G_{(0n01)}\chi^s_{(11)}\chi^v_{(1)}\right) \\
& \quad + t^{8+2n}\left(\chi^G_{(1n01)}\chi^s_{(11)} + \chi^G_{(0n10)}\chi^s_{(20)}\chi^v_{(1)} + \chi^G_{(0n02)}\chi^s_{(01)}\chi^v_{(1)}\right) \\
& \quad - t^{9+2n}\left(\chi^G_{(1n10)}\chi^s_{(20)} + \chi^G_{(1n02)}\chi^s_{(01)} + \chi^G_{(0n11)}\chi^s_{(10)}\chi^v_{(1)}\right) \\
& \quad + t^{10+2n}\left(\chi^G_{(1n11)}\chi^s_{(10)} + \chi^G_{(0n20)}\chi^v_{(1)}\right) - t^{11+2n}\chi^G_{(1n20)}\Bigg),
\end{aligned}
\tag{5.2.16}
$$

which is tested against the general formula (5.1.39) up to $t^{20}$ order. It is the same as (H.20) of [92] after reducing the $Sp(3)_v$ characters by

$$
\chi^v_{(001)} \to \chi^v_{(1)}, \qquad \chi^v_{(010)} \to 1, \qquad \chi^v_{(100)} \to 0, \qquad \chi^v_{(000)} \to 0. \tag{5.2.17}
$$

**SO(10)** We apply our 1-instanton expression (5.1.39) to $SO(10) + N_{\mathbf{s}}\mathbf{S} + N_{\mathbf{c}}\mathbf{C} + N_{\mathbf{v}}\mathbf{V}$ theory with $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) \leq (2,0,2), (1,1,2), (0,2,2), (1,0,4), (0,1,4)$. The relevant flavor symmetry is $U(N_{\mathbf{s}}+N_{\mathbf{c}}) \times Sp(N_{\mathbf{v}})$ because the $SO(10)$ (conjugate) spinor is a complex representation. Since the $SO(10)$ charge conjugation exchanges the spinor and conjugate spinor representations, *i.e.*, $\chi^G_{(00001)} = (\chi^G_{(00010)})^*$, the instanton partition function for $SO(10) + (N_{\mathbf{s}} \mp 1)\mathbf{S} + (N_{\mathbf{c}} \pm 1)\mathbf{C} + N_{\mathbf{v}}\mathbf{V}$ must be identified with that of $SO(10) + N_{\mathbf{s}}\mathbf{S} + N_{\mathbf{c}}\mathbf{C} + N_{\mathbf{v}}\mathbf{V}$ simply by flipping the sign of mass parameters for (conjugate) spinor hypermultiplets:

$$
\begin{aligned}
Z_1^{N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}}}(m^s_{1,\cdots,N_{\mathbf{s}}}; m^c_{1,\cdots,N_{\mathbf{c}}}) &= Z_1^{N_{\mathbf{s}}-1,N_{\mathbf{c}}+1,N_{\mathbf{v}}}(m^s_{1,\cdots,N_{\mathbf{s}}-1}; m^c_{1,\cdots,N_{\mathbf{c}}+1})\big|_{m^c_{N_{\mathbf{c}}+1} = -m^s_{N_{\mathbf{s}}}} \\
&= Z_1^{N_{\mathbf{s}}+1,N_{\mathbf{c}}-1,N_{\mathbf{v}}}(m^s_{1,\cdots,N_{\mathbf{s}}+1}; m^c_{1,\cdots,N_{\mathbf{c}}-1})\big|_{m^s_{N_{\mathbf{s}}+1} = -m^c_{N_{\mathbf{c}}}}.
\end{aligned}
\tag{5.2.18}
$$

This relation is explicitly confirmed in all above cases at 1-instanton order. We may want to compare (5.1.39) with the known 1-instanton partition function of

$SO(10)+1\mathbf{S}+1\mathbf{C}+4\mathbf{V}$ theory, written in (H.21) of [92], after taking relevant mass parameters to infinity. However, (H.21) specifies $\tilde{Z}_1$ only up to $\mathcal{O}(t^5)$, which leaves nothing for comparison once we reduce the mass parameters. Thus the consistency between two expressions can be only weakly tested. For instance, $\tilde{Z}_1$ obtained from (5.1.39) for $SO(10)+N_\mathbf{s}\mathbf{S}+N_\mathbf{c}\mathbf{C}+4\mathbf{V}$ theory with $N_\mathbf{s}+N_\mathbf{c}=2$ is displayed in (A.0.5), which turns out to be trivial upto $t^4$ order.

**SO(12)** The 1-instanton partition function of $SO(12)+1\mathbf{S}+6\mathbf{V}$ theory is written in (H.22) of [92], up to $t^8$ order. It can be compared with our 1-instanton formula (5.1.39) applied to $SO(12)+N_\mathbf{s}\mathbf{S}+N_\mathbf{c}\mathbf{C}+N_\mathbf{v}\mathbf{V}$ theory with $(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v}) \leq (1,0,4)$ or $(0,1,4)$, whose flavor symmetry acting on matter multiplets is $SO(2N_\mathbf{s})_s \times SO(2N_\mathbf{c})_c \times Sp(N_\mathbf{v})_v$. For comparison, we need to appropriately decouple some mass parameters in (H.22) to infinity. It reduces the $Sp(6)_v$ characters in (H.22) to, e.g., the $Sp(4)_v$ irreducible characters as follows:

$$\begin{aligned} &\chi^v_{(000000)} \to 0, && \chi^v_{(100000)} \to 0, && \chi^v_{(010000)} \to 1, \\ &\chi^v_{(001000)} \to \chi^v_{(1000)}, && \chi^v_{(000100)} \to \chi^v_{(0100)}, && \chi^v_{(000001)} \to \chi^v_{(0001)}. \end{aligned} \tag{5.2.19}$$

We explicitly confirmed that (H.22) and (5.1.39) agree up to the given order, for $(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(1,0,4)$. Moreover, we checked that the 1-instanton results $Z_1$ from (5.1.39) for $(N_\mathbf{s},N_\mathbf{c},N_\mathbf{v})=(1,0,N_\mathbf{v})$ and $(0,1,N_\mathbf{v})$ could be interchanged as follows:

$$Z_1^{N_\mathbf{s}=1,N_\mathbf{c}=0,N_\mathbf{v}}(a_1,a_2,a_3,a_4,a_5,a_6) = Z_1^{N_\mathbf{s}=0,N_\mathbf{c}=1,N_\mathbf{v}}(a_1,a_2,a_3,a_4,a_5,-a_6). \tag{5.2.20}$$

**Summary of new results** We have compared so far the solution $Z_1$ of the recursion formulae (5.1.33) with the known 1-instanton partition function for various $SO(N)$ theories with spinor hypermultiplets. The comparison showed consistency for all the examples whose $Z_1$ had been computed [91, 92]. We also collect the character expansion of the 1-instanton partition function (5.1.39) in Appendix A for novel $SO(N)$ theories with spinor matters. See Table 5.2 for the list of character expansions.

| Gauge Group | Hypermultiplets | Equation No. |
|---|---|---|
| $SO(8)$ | $1\mathbf{S} + 1\mathbf{C} + 1\mathbf{V}$ | (5.2.10) |
| $SO(8)$ | $2\mathbf{S} + 2\mathbf{C} + 2\mathbf{V}$ | (5.2.11) |
| $SO(8)$ | $3\mathbf{S} + 1\mathbf{C}$ | (A.0.1) |
| $SO(9)$ | $2\mathbf{S} + 1\mathbf{V}$ | (5.2.16) |
| $SO(10)$ | $2\mathbf{S} + 2\mathbf{V}$ | (A.0.5) |
| $SO(10)$ | $3\mathbf{S}$ | (A.0.7) |
| $SO(11)$ | $1\mathbf{S} + 3\mathbf{V}$ | (A.0.10) |
| $SO(12)$ | $2\mathbf{S}$ | (A.0.11) |
| $SO(12)$ | $1\mathbf{S} + 1\mathbf{C}$ | (A.0.13) |
| $SO(13)$ | $1\mathbf{S} + 1\mathbf{V}$ | (A.0.14) |
| $SO(14)$ | $1\mathbf{S} + 2\mathbf{V}$ | (A.0.15) |

Table 5.2: Character expansion of $SO(N)$ theory with spinor hypermultiplets

### 5.2.3 Theories with an exceptional gauge group

Let us continue to apply the recursion formulae (5.1.33) and the general 1-instanton expression (5.1.39) to study the instanton partition function of exceptional gauge theories. One can find a sufficient number of recursion formulae (5.1.33) to fix the $n$-instanton partition function $Z_n$, if and only if the gauge theory has the following number of fundamental hypermultiplets:

$$\begin{aligned} N_f &\leq 2 &\quad \text{if} \quad & G = G_2, \\ N_f &\leq 2 &\quad \text{if} \quad & G = F_4, \\ N_f + N_{\overline{f}} &\leq 3 &\quad \text{if} \quad & G = E_6, \\ N_f &\leq 2 &\quad \text{if} \quad & G = E_7, \\ &\varnothing &\quad \text{if} \quad & G = E_8. \end{aligned} \tag{5.2.21}$$

Notice that other representations do not appear in the recent classification of 4d $\mathcal{N} = 2$ SCFTs [121] nor 5d SCFTs [34].

| Gauge Group | Hypermultiplets | Equation No. |
|---|---|---|
| $F_4$ | $2\mathbf{F}$ | (5.2.23) |
| $E_6$ | $3\mathbf{F}$ | (A.0.17) |
| $E_7$ | $2\mathbf{F}$ | (A.0.20) |
| $E_8$ | $\emptyset$ | (5.2.25) |

Table 5.3: Character expansion of exceptional gauge theory with fundamental hypermultiplets

We give explicit character expansion of the one instanton partition function in Appendix A. See Table 5.3 for the list of character expansions.

**$\mathbf{G_2}$** A supersymmetric quantum mechanical model was proposed in [91], whose Witten index corresponds to the $n$-instanton partition function of $G_2 + N_f\mathbf{F}$ theory with $N_f \leq 3$. Its index can be written as the following sum over $SU(3)$

colored Young diagrams:

$$Z_n^{\rm YD}=\sum_{|\vec{Y}|=n}\prod_{i=1}^{3}\prod_{s\in Y_i}\frac{2\sinh\left(\phi(s)\right)\,2\sinh\left(\epsilon_+-\phi(s)\right)\,\prod_{l=1}^{N_f}2\sinh(\frac{m_l\pm\phi(s)}{2})}{2\sinh\frac{\epsilon_+-\phi(s)}{2}\prod_{j=1}^{3}2\sinh\frac{E_{ij}}{2}\,2\sinh\frac{E_{ij}-2\epsilon_+}{2}\,2\sinh\frac{\epsilon_+-\phi(s)-a_j}{2}}$$
$$\times\prod_{i\leq j}^{3}\prod_{\substack{s_{i,j}\in Y_{i,j}\\ s_i<s_j}}\frac{2\sinh\frac{\phi(s_i)+\phi(s_j)}{2}\,2\sinh\frac{\phi(s_i)+\phi(s_j)-2\epsilon_+}{2}}{2\sinh\frac{\epsilon_1-\phi(s_i)-\phi(s_j)}{2}\,2\sinh\frac{\epsilon_2-\phi(s_i)-\phi(s_j)}{2}}. \tag{5.2.22}$$

Our 1-instanton formula (5.1.39) agrees with the above expression $Z_1^{\rm YD}$ for all $N_f\leq 2$. Also at two instantons, we explicitly checked that $Z_2^{\rm YD}=Z_2$, where $Z_2$ is the solution of the recursion formulae (5.1.33) with (5.1.47).

$\mathbf{F_4}$ The 1-instanton partition function of $F_4+2\mathbf{F}$ gauge theory is given in (H.31) of [92], which has $Sp(2)_f$ flavor symmetry. In terms of $F_4$ and $Sp(2)_f$ characters,

$$\begin{aligned}\tilde{Z}_1=&\ t^6\chi^f_{(01)}+t^7\chi^f_{(30)}-t^8\left(\chi^G_{(0001)}\chi^f_{(20)}+\chi^G_{(1000)}\right)+t^9\chi^G_{(0010)}\chi^f_{(10)}-t^{10}\chi^G_{(0100)}\\&+\sum_{n=0}^{\infty}\Bigg(t^{8+2n}\chi^G_{(n000)}\chi^f_{(03)}-t^{9+2n}\chi^G_{(n001)}\chi^f_{(12)}+t^{10+2n}\left(\chi^G_{(n010)}\chi^f_{(21)}+\chi^G_{(n002)}\chi^f_{(02)}\right)\\&\qquad-t^{11+2n}\left(\chi^G_{(n100)}\chi^f_{(30)}+\chi^G_{(n011)}\chi^f_{(11)}\right)+t^{12+2n}\left(\chi^G_{(n101)}\chi^f_{(20)}+\chi^G_{(n020)}\chi^f_{(01)}\right)\\&\qquad-t^{13+2n}\chi^G_{(n110)}\chi^f_{(10)}+t^{14+2n}\chi^G_{(n200)}\Bigg).\end{aligned} \tag{5.2.23}$$

We confirmed that our 1-instanton formula (5.1.39) agrees with the above expression up to $t^{15}$ order.

$\mathbf{E_6}$ Let us apply our general 1-instanton expression (5.1.39) to $E_6+N_f\mathbf{F}+N_{\bar{f}}\overline{\mathbf{F}}$ gauge theory with $(N_f,N_{\bar{f}})\leq(3,0),(2,1),(1,2),(0,3)$ whose flavor symmetry is $U(N_f+N_{\bar{f}})$. Since the fundamental and anti-fundamental representations are interchanged by the $E_6$ charge conjugation, their instanton partition functions should be identical upon inverting the sign of relevant mass parameters. We explicitly confirmed that (5.1.39) satisfies the relations

$$\begin{aligned} Z_1^{N_f,N_{\bar f}}(m^f_{1,\cdots,N_f};m^{\bar f}_{1,\cdots,N_{\bar f}}) &= Z_1^{N_f-1,N_{\bar f}+1}(m^f_{1,\cdots,N_f-1};m^{\bar f}_{1,\cdots,N_{\bar f}+1})\big|_{m^{\bar f}_{N_{\bar f}+1}=-m^f_{N_f}} \\ &= Z_1^{N_f+1,N_{\bar f}-1}(m^f_{1,\cdots,N_f+1};m^{\bar f}_{1,\cdots,N_{\bar f}-1})\big|_{m^f_{N_f+1}=-m^{\bar f}_{N_{\bar f}}}, \end{aligned} \tag{5.2.24}$$

in all above cases. Furthermore, $Z_1$ at $(N_f, N_{\bar f}) = (3, 0)$ can be compared with (H.35) of [92] which displays the character expansion up to $t^{11}$ order. We checked their consistency except a sign mistake in the second term of (H.35). The full character expansion of $Z_1$ at $N_f = 3$ and $N_{\bar f} = 0$ is written in (A.0.17), after turning off the $E_6$ Coulomb VEV $\vec a = 0$ for simplicity.

$\mathbf{E_7}$ Our 1-instanton expression (5.1.39) is applicable to $E_7+N_f\mathbf{F}$ gauge theory with $N_f \le 2$, which has $SO(2N_f)$ flavor symmetry. We give the full character expansion of $Z_1$ at $N_f = 2$ in (A.0.20) after setting $\vec a = 0$ to shorten the expression. We also compared the result (5.1.39) applied to the $N_f = 1$ case with (H.40) of [92] and found that they agree up to $t^{280}$ order.

$\mathbf{E_8}$ The (centered) 1-instanton partition function of $E_8$ gauge theory can be written as

$$\tilde Z_1 = \sum_{n=0}^{\infty} t^{29+2n}\chi^{E_8}_{(000000n0)}. \tag{5.2.25}$$

We confirmed that it agrees with our 1-instanton expression (5.1.39) up to $t^{520}$ order. It is actually proven in [90, 100] that the (centered) 1-instanton formula (5.1.39) for any gauge group without matter can be written in terms of the character expression [53, 73, 74]

$$\tilde{Z}_1 = t^{h^\vee - 1} \sum_{n=0}^{\infty} t^{2n} \chi^G_{n\cdot\mathbf{adj}} \,. \tag{5.2.26}$$

### 5.2.4 SU(6) theory with a rank-3 antisymmetric hypermultiplet

Another non-trivial test of our blow-up recursion formulae (5.1.33) is the partition function for 5d $SU(6)$ theory with a hypermultiplet in the rank-3 antisymmetric representation (**TAS**). This theory has can be Higgsed to a theory with $SU(3) \times SU(3)$ gauge symmetry that can be explicitly checked at the level of the partition function.

To have a UV fixed point, 5d $SU(6)$ theories can have up to 2 hypermultiplets in the rank-3 antisymmetric representation [34]. Their type IIB 5-brane configurations were constructed in [93] with/without O5-planes. In particular, 5-brane web diagrams for $SU(6) + \frac{1}{2}\mathbf{TAS}$ and $SU(6) + 1\mathbf{TAS}$ do not contain orientifold planes, so that topological vertex method [60, 98] can be straightforwardly applied to compute their partition functions. In [93], for instance, the partition function of $SU(6)_{\frac{5}{2}} + \frac{1}{2}\mathbf{TAS}$ theory was computed up to two instantons using the topological vertex formalism.

Our blow-up equation (5.1.17) demands all mass parameters to be generically turned on. In particular, we need a mass parameter for the rank-3 antisymmetric hypermultiplet. As one cannot introduce mass for a half-hypermultiplet, let us consider the $SU(6)_3$ theory with a full hypermultiplet in the rank-3 antisymmetric representation ($SU(6)_3 + 1\mathbf{TAS}$). An example for 5-brane web for $SU(6)_3 + 1\mathbf{TAS}$ is depicted in Figure 5.1. It is instructive to see if Figure 5.1 is consistent with the expected prepotential. The effective prepotential on the Coulomb branch of a 5d gauge theory with a gauge group $G$ and matter $f$ in a

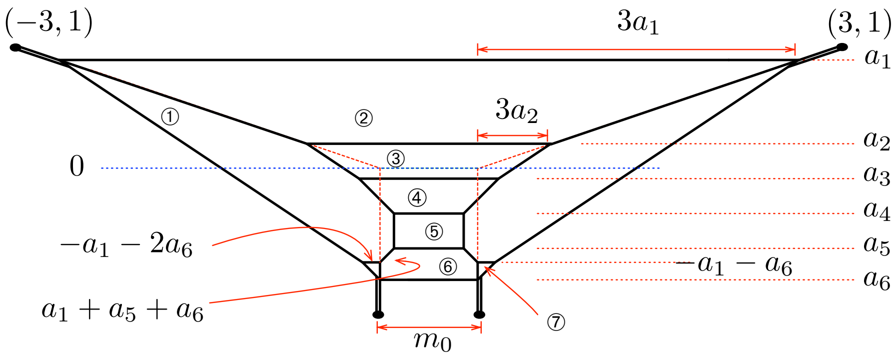



Figure 5.1: A 5-brane web for $SU(6)_3$ theory with one massless hypermultiplet in the rank-3 antisymmetric representation.

representation $\mathbf{R}_f$ is [32]

$$\mathcal{F}(\phi) = \frac{m_0}{2} h_{ij}\phi_i\phi_j + \frac{\kappa}{6} d_{ijk}\phi_i\phi_j\phi_k + \frac{1}{12}\left(\sum_{\vec{\alpha}\in\Delta}\left|\vec{\alpha}\cdot\vec{\phi}\right|^3 - \sum_f \sum_{\vec{\omega}\in\mathbf{R}_f}\left|\vec{\omega}\cdot\vec{\phi} + m_f\right|^3\right). \tag{5.2.27}$$

Here, $m_0$ is the inverse of the gauge coupling squared, $\kappa$ is the Chern-Simons level and $m_f$ is a mass parameter for the matter $f$. $\vec{\alpha}$ is a root of the Lie algebra $\mathfrak{g}$ associated to $G$ and $\vec{\omega}$ is a weight of the representation $\mathbf{R}_f$ of $\mathfrak{g}$. We also defined $h_{ij} = \mathrm{Tr}(T_iT_j), d_{ijk} = \frac{1}{2}\mathrm{Tr}\left(T_i\{T_j, T_k\}\right)$ where $T_i$ are the Cartan generators of the Lie algebra $\mathfrak{g}$. With the Coulomb branch moduli assigned in Figure 5.1 and the identification of Weyl chamber for the Coulomb VEV ($a_1 \geq a_2 \geq \cdots \geq a_6$, $\sum_{i=1}^{6} a_i = 0$),

$$a_1 = \phi_1, \quad a_2 = \phi_2 - \phi_1, \quad a_3 = \phi_3 - \phi_2, \quad a_4 = \phi_4 - \phi_3, \quad a_5 = \phi_5 - \phi_4, \quad a_6 = -\phi_5, \tag{5.2.28}$$

one finds that the prepotential for $SU(6)_3$ with one *massless* rank-3 antisym-

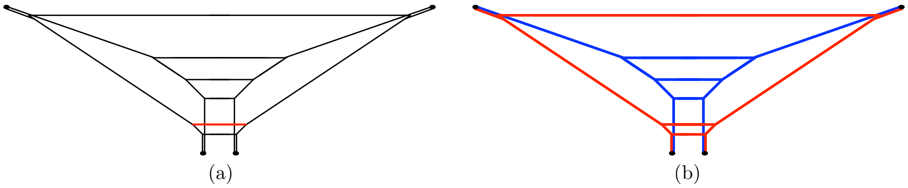


Figure 5.2: (a) A Higgsing of $SU(6)_3 + 1\mathbf{TAS}$ into two $SU(3)_3$ theories by aligning internal D5-branes in red. (a) Two different $SU(3)_3$ theories are painted in blue and red, respectively.

metric matter takes the form of

$$\begin{aligned}\mathcal{F}_{SU(6)_3+1\mathbf{TAS}} = m_0\big(&\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2+\phi_5^2-\phi_1\phi_2-\phi_2\phi_3-\phi_3\phi_4-\phi_4\phi_5\big)\\ &+\frac{\phi_1^3}{3}+\frac{4\phi_2^3}{3}+\frac{4\phi_3^3}{3}+\frac{4\phi_4^3}{3}+\frac{4\phi_5^3}{3}+4\phi_1^2\phi_2-5\phi_1\phi_2^2\\ &-2\phi_1\big(\phi_3^2+\phi_4^2+\phi_5^2\big)+\phi_2^2\phi_3-2\phi_2\phi_3^2-\phi_3\phi_4^2-\phi_4^2\phi_5\\ &+2\phi_1\phi_2\phi_3+2\phi_1\phi_3\phi_4+2\phi_1\phi_4\phi_5.\end{aligned} \tag{5.2.29}$$

One can easily see that the monopole string tensions $T_i = \partial\mathcal{F}/\partial\phi_i$ computed from the above prepotential (5.2.29) agree with the areas of the compact faces of the 5-brane web, i.e.,

$$T_1 = ① + 2\times ②,\quad T_2 = ③,\quad T_3 = ④,\quad T_4 = ⑤,\quad T_5 = ⑥ + 2\times ⑦, \tag{5.2.30}$$

where the encircled numbers represent the area of apparent faces in Figure 5.1. This shows that Figure 5.1 is indeed consistent with the prepotential of $SU(6)_3 + 1\mathbf{TAS}$ gauge theory.

Notice that this 5-brane web for $SU(6)_3 + 1\mathbf{TAS}$ suggests an intriguing Higgsing of the theory, which is the Higgsing of $SU(6)$ theory with one rank-3 antisymmetric hyper into two disjoint $SU(3)$ theories. It can be achieved by setting the Coulomb branch parameters as

$$a_5 = -a_1 - a_6, \qquad \text{or equivalently} \qquad \phi_4 = \phi_1. \tag{5.2.31}$$

This tuning of the parameters, of course, reduces dimension of the Coulomb branch by one and also opens up a Higgs branch in such a way that the 5-brane web in Figure 5.1 becomes 5-brane web in Figure 5.2(a) where the D5-branes on the upper edges of ⑥ and ⑦ are aligned and joint to become a single D5-brane denoted red in Figure 5.2(a). The resulting configuration is then a 5-brane configuration for two pure $SU(3)_3$ theories that are on top of each other, as shown in Figure 5.2(b). This is a 5-brane realization of Higgsing $SU(6)_3 + 1\mathbf{TAS}$ theory into two pure $SU(3)_3$ theories. It follows that under this Higgsing, the prepotential for $SU(6)_3 + 1\mathbf{TAS}$ (5.2.29) theory reduces to a sum of prepotentials for two disjoint pure $SU(3)_3$ theories:

$$\mathcal{F}_{SU(6)_3+1\mathbf{TAS}}\Big|_{a_1+a_5+a_6=0} \to \mathcal{F}_{SU(3)_3}(m_0, a_1, a_5, a_6) + \mathcal{F}_{SU(3)_3}(m_0, a_2, a_3, a_4). \tag{5.2.32}$$

This in turn implies that under this Higgsing, the partition function for $SU(6)_3+1\mathbf{TAS}$ should be expressed as a product of the partition functions of two pure $SU(3)_3$ theories:

$$\mathcal{Z}^{SU(6)_3+1\mathbf{TAS}}\big|_{\text{Higgsing}} \;\to\; \mathcal{Z}^{SU(3)_3}(q, A_1, A_5, A_6)\, \mathcal{Z}^{SU(3)_3}(q, A_2, A_3, A_4)\, Z_{\text{extra}}(q)\,, \tag{5.2.33}$$

where the parameters $q$ and $A_i$ are the Kähler parameters for instanton and Coulomb branch parameters, and $Z_{\text{extra}}(q)$ represents the overall extra terms that do not explicitly depend on the Coulomb branch moduli, which would correspond to a new decoupled mode appearing in Figure 5.2. In what follows, we explicitly compute the partition function for $SU(6)_3 + 1\mathbf{TAS}$ based on the 5-brane web and compare it with our general 1-instanton formula (5.1.39). At two instantons, we will consider this Higgsing as a consistency check of our solution $Z_2$ obtained from the blowup recursion formulae (5.1.33).

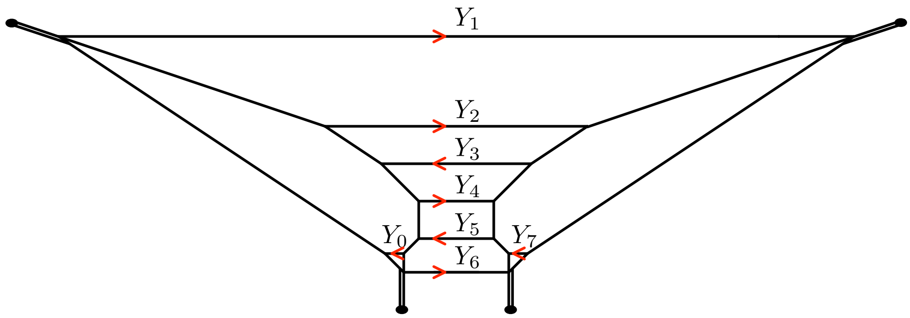



Figure 5.3: A labeling of Young diagrams assigned to the horizontal edges of Figure 5.1.

To compute the instanton partition function based on the 5-brane web for $SU(6)_3 + 1\mathbf{TAS}$ given in Figure 5.1, we assign the Young diagrams $Y_i$ to each horizontal edge of the web diagram as shown in Figure 5.3 and use the topological vertex method. For convenience, we restrict ourselves to the unrefined case where $2\epsilon_+ = \epsilon_1 + \epsilon_2 = 0$. (See also a similar calculation done in [93].) As the web diagram in Figure 5.1 is left-right symmetric, it is convenient to split the web diagram to the left and right parts and glue them later to obtain the full partition function. Let us introduce the following fugacity variables to express the partition function.

$$A_i \equiv e^{-a_i} \quad \text{for} \quad i = 1, \cdots, 6, \qquad g \equiv \sqrt{p_1/p_2} = e^{-\epsilon_-}, \tag{5.2.34}$$

in which the $SU(6)$ traceless condition $\prod_{i=1}^{6} A_i = 1$ is assumed. Applying the topological vertex formalism [60], we find that

$$\begin{aligned} \mathcal{Z} = \sum_{(Y_1,\cdots,Y_6)} & q^{\sum_{i=1}^{6}|Y_i|}(-A_1^6)^{|Y_1|}(-A_2^6)^{|Y_2|}(-A_2^2A_3^4)^{|Y_3|}(-A_2^2A_3^2A_4^2)^{|Y_4|+|Y_5|} \\ & \times f_{Y_1}(g)^5 f_{Y_2}(g)^5 f_{Y_3}(g)^3 f_{Y_4}(g) f_{Y_5}(g)^{-1} f_{Y_6}(g)^2 Z_{\text{left}}(\vec{Y}) Z_{\text{right}}(\vec{Y}), \end{aligned} \tag{5.2.35}$$

where $\vec{Y} = (Y_1, Y_2, Y_3, Y_4, Y_5, Y_6)$. The left/right factor $Z_{\text{left}}(\vec{Y})/Z_{\text{right}}(\vec{Y})$ can be written as

$$
\begin{aligned}
Z_{\text{left}}(\vec{Y}) = Z_{\text{right}}(\vec{Y}) = \sum_{Y'} & (-A_1{}^{-1}A_6{}^{-2})^{|Y'|} g^{\frac{||Y'^t||^2+||Y'||^2}{2}} \tilde{Z}^2_{Y'} f^2_{Y'}(g) \prod_{i=1}^{6} g^{\frac{||Y_i||^2}{2}} \tilde{Z}_{Y_i} \\
& \times R^{-1}_{Y_1 Y_6^t}(A_1 A_6{}^{-1})\, R^{-1}_{Y' Y_6^t}(A_1{}^{-1}A_6{}^{-2})\, R^{-1}_{Y_1 Y'^t}(A_1^2 A_6) \\
& \times \prod_{2\le i<j\le 5} R^{-1}_{Y_i Y_j^t}(A_i A_j{}^{-1}) \prod_{i=2}^{5} R_{Y'^t Y_i}(A_1 A_i A_6)
\end{aligned}
\tag{5.2.36}
$$

in which the dummy variable $Y'$ should be interpreted as $Y_0$ for $Z_{\text{left}}(\vec{Y})$ and $Y_7$ for $Z_{\text{right}}(\vec{Y})$. Here, for a Young diagram $\lambda = (\lambda_1, \lambda_2, \cdots)$ and its transpose $\lambda^t$,

$$
|\lambda| = \sum_i \lambda_i, \qquad ||\lambda||^2 = \sum_i \lambda_i^2, \qquad \tilde{Z}_\lambda = \prod_{(i,j)\in\lambda} \frac{1}{1 - g^{\lambda_i + \lambda_j^t - i - j + 1}}. \tag{5.2.37}
$$

The framing factor $f_\lambda(g)$ is defined by

$$
f_\lambda(g) = (-1)^{|\lambda|} g^{\frac{1}{2}(g^{||\lambda^t||^2 - ||\lambda||^2})}. \tag{5.2.38}
$$

And also, $R_{\lambda\mu}(Q) = R_{\mu\lambda}(Q)$ is defined by

$$
R_{\lambda\mu}(Q) = \text{PE}\left[-\frac{g}{(1-g)^2} Q\right] \times N_{\lambda^t\mu}(Q), \tag{5.2.39}
$$

with PE representing the Plethystic exponential (5.1.25) and

$$
N_{\lambda\mu}(Q) = \prod_{(i,j)\in\lambda} \left(1 - Q g^{\lambda_i + \mu_j^t - i - j + 1}\right) \prod_{(i,j)\in\mu} \left(1 - Q g^{-\lambda_j^t - \mu_i + i + j - 1}\right). \tag{5.2.40}
$$

Recall that the Nekrasov partition function is expressed as the following weighted sum:

$$
\mathcal{Z} = Z_{\text{pert}} \cdot \left(1 + \sum_{k=1}^{\infty} q^k Z_k\right), \tag{5.2.41}
$$

where $Z_{\text{pert}}$ is the perturbative partition function, while $Z_k$ stands for the $k$-instanton partition function. The perturbative part of the partition function $Z_{\text{pert}}$ comes from the summand of (5.2.35) at empty Young diagrams, i.e., $(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6) = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$. It is given by

$$\begin{aligned}
Z_{\text{pert}} =& Z_{\text{left}}(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) Z_{\text{right}}(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) \\
=& \mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\frac{A_1}{A_6} + \frac{1}{A_1 A_6^2} + A_1^2 A_6 + \sum_{2\le i<j\le 5} \frac{A_i}{A_j} - \sum_{i=2}^{5} A_1 A_i A_6\Big)\right] \\
& \times \Bigg( \sum_{Y'} (-A_1{}^{-1} A_6{}^{-2})^{|Y'|}\, g^{\frac{\|Y'^t\|^2 + \|Y'\|^2}{2}}\, \tilde{Z}_{Y'}(g)^2 f_{Y'}^2(g) \\
& \qquad N^{-1}_{Y'^t\emptyset}(A_1{}^{-1} A_6{}^{-2}) N^{-1}_{Y'\emptyset}(A_1^2 A_6) \textstyle\prod_{i=2}^{5} N_{Y'\emptyset}(A_1 A_i A_6)\Bigg)^2,
\end{aligned} \tag{5.2.42}$$

where the last two lines can be combined into the following closed-form expression:

$$\mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\sum_{i=2}^{5} \frac{A_1}{A_i} + \sum_{i=2}^{5} \frac{A_i}{A_6} - \frac{1}{A_1 A_6^2} - A_1^2 A_6 - \sum_{2\le i<j\le 5} A_1 A_i A_j + \mathcal{O}(A_1^6)\Big)\right]. \tag{5.2.43}$$

We note here that when performing the Young diagram sum over $Y'$ in (5.2.42) to compute the $Z_{\text{pert}}$, we expand (5.2.42) in terms of $A_1$ and, by $\mathcal{O}(A_1^6)$, we mean that the obtained result is explicitly compared up to $\mathcal{O}(A_1^6)$. As it is very unlikely that there will be a new term which suddenly appears in higher orders than 6 in $A_1$, we believe that there are no further terms for $\mathcal{O}(A_1^6)$. It is clear then that (5.2.42) is manifestly consistent with the equivariant index [24] for 5d $SU(6)$ gauge theory with a hypermultiplet in the rank-3 antisymmetric representation, i.e.,

$$Z_{\text{pert}} = \mathrm{PE}\left[\frac{2g}{(1-g)^2}\Big(\sum_{1\le i<j\le 6} \frac{A_i}{A_j} - \sum_{2\le i<j\le 6} A_1 A_i A_j\Big)\right]. \tag{5.2.44}$$

The 1-instanton partition function $Z_1$ can be obtained from the summands of (5.2.35) at Young diagrams satisfying $\sum_{i=1}^{6}|Y_i| = 1$. There are 6 different profiles of Young diagrams. The configuration $|Y_i| = 1$ and $Y_{j\neq i} = \varnothing$ contribute to $Z_1$ as

$$+\frac{g}{(1-g)^2}\frac{A_i^6}{\prod_{j\neq i}(A_i - A_j)^2}\Big(-A_i\sum_{j\neq i}A_j + \sum_{j\neq i}\frac{1}{A_j} - \frac{1}{A_i} + A_i^2\Big)^2. \tag{5.2.45}$$

Summing over all six contributions, one finds

$$Z_1 = \sum_{i=1}^{6}\frac{g}{(1-g)^2}\frac{A_i^6}{\prod_{j\neq i}(A_i - A_j)^2}\Big(-A_i\sum_{j\neq i}A_j + \sum_{j\neq i}\frac{1}{A_j} - \frac{1}{A_i} + A_i^2\Big)^2. \tag{5.2.46}$$

which is in agreement with our general 1-instanton formula (5.1.39).

We checked that upon imposing the Higgsing condition (5.2.33), i.e., $a_1 + a_5 + a_6 = 0$ and $a_2 + a_3 + a_4 = 0$, the 1-loop contribution (5.2.44) can be factorized into a product of two $SU(3)$ vector multiplet indices (5.1.23). We also confirmed that the instanton corrections $Z_1$ and $Z_2$ obtained from the blowup recursion formulae (5.1.33) with (5.1.50) become

$$\begin{aligned}
Z_1^{SU(6)_3+1\mathbf{TAS}}\big|_{\text{Higgsing}} &\to Z_1^{SU(3)_3}(A_1,A_5,A_6) + Z_1^{SU(3)_3}(A_2,A_3,A_4),\\
Z_2^{SU(6)_3+1\mathbf{TAS}}\big|_{\text{Higgsing}} &\to Z_2^{SU(3)_3}(A_1,A_5,A_6) + Z_2^{SU(3)_3}(A_2,A_3,A_4)\\
&\quad + Z_1^{SU(3)_3}(A_1,A_5,A_6)\cdot Z_1^{SU(3)_3}(A_2,A_3,A_4),
\end{aligned} \tag{5.2.47}$$

which satisfy the expected Higgsing relation (5.2.33). Here, $Z_n^{SU(3)_3}$ is the Young diagram formula (5.2.1) which includes the Coulomb VEV independent contribution $Z_{\text{extra}}(q)$.

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# Appendix A

# One-instanton partition functions

This appendix collects the character expansion of the 1-instanton partition function $Z_1$ for a variety of 5d $\mathcal{N}=1$ gauge theories. For simplicity, we display the $\tilde{Z}_1 \equiv (2\sinh\frac{\epsilon_{1,2}}{2})\cdot Z_1$ which takes off the center-of-mass factor. They are written in terms of irreducible characters $\chi^S_{\mathbf{R}}$, whose superscript $S \in \{G, v, s, c, f, \bar{f}\}$ indicates the gauge symmetry ($G$) or the flavor symmetry acting on the vector ($v$), spinor ($s$), conjugate spinor ($c$), fundamental ($f$), or anti-fundamental ($\bar{f}$) hypermultiplets. The representation $\mathbf{R}$ of an irreducible character $\chi^S_{\mathbf{R}}$ is specified by its Dynkin label.[1] An irreducible character for the flavor symmetry is assumed to be in the orthogonal basis, such that it can be consistent with the mass parameters $m_\ell$ introduced in Section 5.1. We will often distinguish the mass parameters by the superscript $S \in \{s, c, v, f, \bar{f}\}$ according to the matter representation.

[1] In this paper, we follow the convention of LieART [36] to denote the Dynkin label of a representation $\mathbf{R}$.

**SO(8)** The flavor symmetry acting on $N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ matter multiplets is given by $Sp(N_{\mathbf{s}})_s \times Sp(N_{\mathbf{c}})_c \times Sp(N_{\mathbf{v}})_v$. For $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (3,1,0)$, the character expansion of the 1-instanton result $\tilde{Z}_1$ is

$$\begin{aligned}\tilde{Z}_1 = \sum_{n=0}^{\infty} \Big( & t^{5+2n}\chi^G_{(0n00)}\chi^s_{(001)}\chi^v_{(1)} - t^{6+2n}(\chi^G_{(0n01)}\chi^s_{(010)}\chi^v_{(1)} + \chi^G_{(1n00)}\chi^s_{(001)}) \\ & + t^{7+2n}(\chi^G_{(1n01)}\chi^s_{(010)} + \chi^G_{(0n02)}\chi^s_{(100)}\chi^v_{(1)}) \\ & - t^{8+2n}(\chi^G_{(1n02)}\chi^s_{(100)} + \chi^G_{(0n03)}\chi^v_{(1)}) + t^{9+2n}\chi^G_{(1n03)}\Big),\end{aligned} \tag{A.0.1}$$

which was compared with the closed-form expression (5.1.39) up to $t^{20}$ order. We checked that the 1-instanton partition functions $Z_1$ from (5.1.39) for $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (3,1,0)$ and $(1,3,0)$ could be interchanged as follows:

$$Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(1,3,0)}(a_1,a_2,a_3,a_4) = Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(3,1,0)}(a_1,a_2,a_3,-a_4). \tag{A.0.2}$$

The $SO(8)$ triality (5.2.12) was also confirmed as in Section 5.2.2. Namely, we found that

$$\begin{aligned}Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(3,1,0)}(\vec{a},\epsilon_1,\epsilon_2;\vec{m}^s,\vec{m}^c,0) &= Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(0,3,1)}(\vec{a}',\epsilon_1,\epsilon_2;0,\vec{m}^s,\vec{m}^c)|_{\vec{a}'\to\vec{a}} \\ &= Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(1,0,3)}(\vec{a}'',\epsilon_1,\epsilon_2;\vec{m}^c,0,\vec{m}^s)|_{\vec{a}''\to\vec{a}},\end{aligned} \tag{A.0.3}$$

$$\begin{aligned}Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(1,3,0)}(\vec{a},\epsilon_1,\epsilon_2;\vec{m}^s,\vec{m}^c,0) &= Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(0,1,3)}(\vec{a}',\epsilon_1,\epsilon_2;0,\vec{m}^s,\vec{m}^c)|_{\vec{a}'\to\vec{a}} \\ &= Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(3,0,1)}(\vec{a}'',\epsilon_1,\epsilon_2;\vec{m}^c,0,\vec{m}^s)|_{\vec{a}''\to\vec{a}}.\end{aligned} \tag{A.0.4}$$

The character expansion for other $SO(8)$ theories with less number of hypermultiplets can be obtained from (A.0.1) by decoupling some mass parameters to infinity. It was checked that the general 1-instanton formula (5.1.39) agrees with that.

**SO(10)** The flavor symmetry acting on $N_{\mathbf{s}}\mathbf{S} + N_{\mathbf{c}}\mathbf{C} + N_{\mathbf{v}}\mathbf{V}$ hypermultiplets is $U(N_{\mathbf{s}} + N_{\mathbf{c}})_s \times Sp(N_{\mathbf{v}})_v$, reflecting that the $SO(10)$ (conjugate) spinor representation is complex. For $N_{\mathbf{s}} + N_{\mathbf{c}} = 2$ and $N_{\mathbf{v}} = 2$, the character expansion of $\tilde{Z}_1$ is given by

$$
\begin{aligned}
\tilde{Z}_1 = {} & t^5(\chi^s_{(2)_0} + \chi^v_{(01)}) + t^6(\chi^s_{(2)_{-2}} + \chi^s_{(2)_2}) - t^7(\chi^G_{(00001)}\chi^s_{(1)_{-1}}\chi^v_{(10)} + \chi^G_{(00010)}\chi^s_{(1)_1}\chi^v_{(10)} + \chi^G_{(01000} \\
& + \chi^G_{(10000)}(\chi^s_{(2)_{-2}} + \chi^s_{(2)_2})) + t^8(\chi^G_{(00100)}\chi^v_{(10)} + \chi^G_{(10001)}\chi^s_{(1)_{-1}} + \chi^G_{(10010)}\chi^s_{(1)_1}) - t^9\chi^G_{(10100)} \\
& + \textstyle\sum_{n=0}^{\infty} \Big(t^{7+2n}\chi^G_{(0n000)}(\chi^s_{(0)_{-4}} + \chi^s_{(0)_4} + \chi^s_{(4)_0})\chi^v_{(01)} - t^{8+2n}(\chi^G_{(0n001)}(\chi^s_{(1)_{-3}} + \chi^s_{(3)_1})\chi^v_{(01)} \\
& \qquad\qquad + \chi^G_{(0n010)}(\chi^s_{(1)_3} + \chi^s_{(3)_{-1}})\chi^v_{(01)} + \chi^G_{(1n000)}(\chi^s_{(0)_{-4}} + \chi^s_{(0)_4} + \chi^s_{(4)_0})\chi^v_{(10)}) \\
& \quad + t^{9+2n}(\chi^G_{(0n100)}(\chi^s_{(2)_{-2}} + \chi^s_{(2)_2})\chi^v_{(01)} + \chi^G_{(0n002)}\chi^s_{(0)_{-2}}\chi^v_{(01)} + \chi^G_{(0n020)}\chi^s_{(0)_2}\chi^v_{(01)} \\
& \qquad\qquad + \chi^G_{(0n011)}\chi^s_{(2)_0}\chi^v_{(01)} + \chi^G_{(1n001)}(\chi^s_{(1)_{-3}} + \chi^s_{(3)_1})\chi^v_{(10)} + \chi^G_{(1n010)}(\chi^s_{(1)_3} + \chi^s_{(3)_{-1}})\chi \\
& \qquad\qquad + \chi^G_{(2n000)}(\chi^s_{(0)_{-4}} + \chi^s_{(0)_4} + \chi^s_{(4)_0})) \\
& \quad - t^{10+2n}(\chi^G_{(0n101)}\chi^s_{(1)_{-1}}\chi^v_{(01)} + \chi^G_{(0n110)}\chi^s_{(1)_1}\chi^v_{(01)} + \chi^G_{(1n100)}(\chi^s_{(2)_{-2}} + \chi^s_{(2)_2})\chi^v_{(10)} \\
& \qquad\qquad + \chi^G_{(1n002)}\chi^s_{(0)_{-2}}\chi^v_{(10)} + \chi^G_{(1n020)}\chi^s_{(0)_2}\chi^v_{(10)} + \chi^G_{(1n011)}\chi^s_{(2)_0}\chi^v_{(10)} \\
& \qquad\qquad + \chi^G_{(2n001)}(\chi^s_{(1)_{-3}} + \chi^s_{(3)_1}) + \chi^G_{(2n010)}(\chi^s_{(1)_3} + \chi^s_{(3)_{-1}}) \\
& \quad + t^{11+2n}(\chi^G_{(1n200)}\chi^v_{(01)} + \chi^G_{(1n101)}\chi^s_{(1)_{-1}}\chi^v_{(10)} + \chi^G_{(1n110)}\chi^s_{(1)_1}\chi^v_{(10)} + \chi^G_{(2n100)}(\chi^s_{(2)_{-2}} + \chi \\
& \qquad\qquad + \chi^G_{(2n002)}\chi^s_{(0)_{-2}} + \chi^G_{(2n020)}\chi^s_{(0)_2} + \chi^G_{(2n011)}\chi^s_{(2)_0}) \\
& \quad - t^{12+2n}(\chi^G_{(1n200)}\chi^v_{(10)} + \chi^G_{(2n101)}\chi^s_{(1)_{-1}} + \chi^G_{(2n110)}\chi^s_{(1)_1}) + t^{13+2n}\chi^G_{(2n200)}\Big).
\end{aligned}
\tag{A.0.5}
$$

where the $U(2)$ character $\chi^s_{(j)_b}$ is defined as (with $y_{s,i} \equiv e^{-m^s_i}$ and $y_{c,i} \equiv e^{-m^c_i}$ understood)

$$
\chi^s_{(j)_b} = \begin{cases} (y_{s,1}\, y_{s,2})^{b/2} \cdot \sum_{a=0}^{j} \big(y_{s,1}/y_{s,2}\big)^{-j/2+a} & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (2,0), \\ (y_{s,1}/y_{c,1})^{b/2} \cdot \sum_{a=0}^{j} \big(y_{s,1}\, y_{c,1}\big)^{-j/2+a} & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (1,1), \\ (y_{c,1}\, y_{c,2})^{-b/2} \cdot \sum_{a=0}^{j} \big(y_{c,1}/y_{c,2}\big)^{-j/2+a} & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (0,2). \end{cases}
\tag{A.0.6}
$$

Similarly, for $N_{\mathbf{s}} + N_{\mathbf{c}} = 3$ and $N_{\mathbf{v}} = 0$, the character expansion of $\tilde{Z}_1$ is given by

$$
\begin{aligned}
\tilde{Z}_1 = & \ t^5(\chi^G_{(10000)} + \chi^s_{(02)_{-2}} + \chi^s_{(20)_2}) - t^6(\chi^s_{(01)_{-1}} + \chi^s_{(10)_1}) + t^7\chi^G_{(00100)} \\
& + \textstyle\sum_{n=0}^{\infty} \Big( t^{7+2n}(\chi^G_{(0n000)}(\chi^s_{(00)_{-6}} + \chi^s_{(00)_6} + \chi^s_{(40)_{-2}} + \chi^s_{(04)_2})) \\
& \qquad - t^{8+2n}(\chi^G_{(0n001)}(\chi^s_{(10)_{-5}} + \chi^s_{(03)_3} + \chi^s_{(31)_{-1}}) + \chi^G_{(0n010)}(\chi^s_{(01)_5} + \chi^s_{(30)_{-3}} + \chi^s_{(13)_1})) \\
& \qquad + t^{9+2n}(\chi^G_{(0n100)}(\chi^s_{(20)_{-4}} + \chi^s_{(02)_4} + \chi^s_{(22)_0}) + \chi^G_{(0n011)}(\chi^s_{(21)_{-2}} + \chi^s_{(12)_2}) \\
& \qquad\qquad + \chi^G_{(0n002)}(\chi^s_{(01)_{-4}} + \chi^s_{(30)_0}) + \chi^G_{(0n020)}(\chi^s_{(10)_4} + \chi^s_{(03)_0}))
\end{aligned}
\tag{A.0.7}
$$

$$
\begin{aligned}
& \qquad - t^{10+2n}(\chi^G_{(0n101)}(\chi^s_{(11)_{-3}} + \chi^s_{(21)_1}) + \chi^G_{(0n110)}(\chi^s_{(11)_3} + \chi^s_{(12)_{-1}}) \\
& \qquad\qquad + \chi^G_{(0n003)}\chi^s_{(00)_{-3}} + \chi^G_{(0n030)}\chi^s_{(00)_3} + \chi^G_{(0n012)}\chi^s_{(20)_{-1}} + \chi^G_{(0n021)}\chi^s_{(02)_1}) \\
& \qquad + t^{11+2n}(\chi^G_{(0n200)}(\chi^s_{(02)_{-2}} + \chi^s_{(20)_2}) + \chi^G_{(0n102)}\chi^s_{(10)_{-2}} + \chi^G_{(0n120)}\chi^s_{(01)_2} + \chi^G_{(0n111)}\chi^s_{(11)_0} \\
& \qquad - t^{12+2n}(\chi^G_{(0n201)}\chi^s_{(01)_{-1}} + \chi^G_{(0n210)}\chi^s_{(10)_1}) + t^{13+2n}\chi_{(0n300)}\Big),
\end{aligned}
$$

where the $U(3)$ character $\chi^s_{(mn)_c}$ is defined as

$$
\chi^s_{(mn)_c} = (w_1w_2w_3)^{\frac{c-m+n}{3}} \left( \sum_{\substack{1\leq i_1\leq\cdots\leq i_m\leq 3 \\ 1\leq j_1\leq\cdots\leq j_n\leq 3}} \frac{w_{i_1}\cdots w_{i_m}}{w_{j_1}\cdots w_{j_n}} - \sum_{\substack{1\leq i_1\leq\cdots\leq i_{m-1}\leq 3 \\ 1\leq j_1\leq\cdots\leq j_{n-1}\leq 3}} \frac{w_{i_1}\cdots w_{i_{m-1}}}{w_{j_1}\cdots w_{j_{n-1}}} \right),
\tag{A.0.8}
$$

with

$$
(w_1, w_2, w_3) = \begin{cases} (y_{s,1}, y_{s,2}, y_{s,3}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (3,0), \\ (y_{s,1}, y_{s,2}, y^{-1}_{c,1}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (2,1), \\ (y_{s,1}, y^{-1}_{c,1}, y^{-1}_{c,2}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (1,2), \\ (y^{-1}_{c,1}, y^{-1}_{c,2}, y^{-1}_{c,3}) & \text{for} \quad (N_{\mathbf{s}}, N_{\mathbf{c}}) = (0,3). \end{cases}
\tag{A.0.9}
$$

Again, (A.0.5) and (A.0.7) was tested against the closed-form expression (5.1.39) up to $t^{20}$ order.

**SO(11)** The flavor symmetry acting on $N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{v}}\mathbf{V}$ hypermultiplets is $SO(2N_{\mathbf{s}})_s\times Sp(N_{\mathbf{v}})_v$. For $N_{\mathbf{s}}=1$ and $N_{\mathbf{v}}=3$, the character expansion of $\tilde{Z}_1$ can be written as

$$
\begin{aligned}
\tilde{Z}_1 = {} & t^5 + t^6(\chi^v_{(001)} + (y_s^2+y_s^{-2})\chi^v_{(100)}) + t^7((y_s^2+y_s^{-2}+1)\chi^v_{(010)} - (y_s^2+y_s^{-2})\chi^G_{(10000)}) \\
& - t^8((y_s+y_s^1)\chi^G_{(00001)}\chi^v_{(010)} + (y_s^2+y_s^{-2}+1)\chi^G_{(10000)}\chi^v_{(100)} + \chi^G_{(01000)}\chi^v_{(100)}) \\
& + t^9(\chi^G_{(00100)}\chi^v_{(010)} + \chi^G_{(10001)}(y_s+y_s^{-1})\chi^v_{(100)} + \chi^G_{(20000)}(y_s^2+y_s^{-2}+1) + \chi^G_{(11000)}) \\
& - t^{10}(\chi^G_{(10100)}\chi^v_{(100)} + \chi^G_{(20001)}(y_s+y_s^{-1})) + t^{11}\chi^G_{(20100)} \qquad \text{(A.0.10)} \\
& + \sum_{n=0}^{\infty}\Big(t^{8+2n}(\chi^G_{(0n000)}(y_s^4+y_s^{-4}+1)\chi^v_{(001)}) \\
& \qquad - t^{9+2n}(\chi^G_{(0n001)}(y_s^3+y_s^{-3})\chi^v_{(001)} + \chi^G_{(0n001)}(y_s+y_s^{-1})\chi^v_{(001)} + \chi^G_{(1n000)}(y_s^4+y_s^{-4}+1) \\
& \qquad + t^{10+2n}(\chi^G_{(0n010)}(y_s^2+y_s^{-2})\chi^v_{(001)} + \chi^G_{(0n100)}(y_s^2+y_s^{-2}+1)\chi^v_{(001)} + \chi^G_{(0n002)}\chi^v_{(001)} \\
& \qquad\qquad + \chi^G_{(1n001)}(y_s^3+y_s+y_s^{-1}+y_s^{-3})\chi^v_{(010)} + \chi^G_{(2n000)}(y_s^4+y_s^{-4}+1)\chi^v_{(100)}) \\
& \qquad - t^{11+2n}(\chi^G_{(0n101)}(y_s+y_s^{-1})\chi^v_{(001)} + \chi^G_{(1n100)}(y_s^2+y_s^{-2}+1)\chi^v_{(010)} + \chi^G_{(1n010)}(y_s^2+y_s^{-2} \\
& \qquad\qquad + \chi^G_{(1n002)}\chi^v_{(010)} + \chi^G_{(2n001)}(y_s^3+y_s+y_s^{-1}+y_s^{-3})\chi^v_{(100)} + \chi^G_{(3n000)}(y_s^4+y_s^{-4}+ \\
& \qquad + t^{12+2n}(\chi^G_{(0n200)}\chi^v_{(001)} + \chi^G_{(1n101)}(y_s+y_s^{-1})\chi^v_{(010)} + \chi^G_{(2n100)}(y_s^2+y_s^{-2}+1)\chi^v_{(100)} \\
& \qquad\qquad + \chi^G_{(2n010)}(y_s^2+y_s^{-2})\chi^v_{(100)} + \chi^G_{(2n002)}\chi^v_{(100)} + \chi^G_{(3n001)}(y_s^3+y_s+y_s^{-1}+y_s^{-3})) \\
& \qquad - t^{13+2n}(\chi^G_{(1n200)}\chi^v_{(010)} + \chi^G_{(2n101)}(y_s+y_s^{-1})\chi^v_{(100)} + \chi^G_{(3n100)}(y_s^2+y_s^{-2}+1) \\
& \qquad\qquad + \chi^G_{(3n010)}(y_s^2+y_s^{-2}) + \chi^G_{(3n002)}) \\
& \qquad + t^{14+2n}(\chi^G_{2n200)}\chi^v_{(100)} + \chi^G_{(3n101)}(y_s+y_s^{-1})) - t^{15+2n}\chi^G_{(3n200)}\Big).
\end{aligned}
$$

which was compared with the closed-form expression (5.1.39) up to $t^{20}$ order.

**SO(12)** The flavor symmetry acting on $N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ hypermultiplets is $SO(2N_{\mathbf{s}})_s\times SO(2N_{\mathbf{c}})_c\times Sp(N_{\mathbf{v}})_v$. Here we turn off the Coulomb VEV $\vec{a}=0$ for simplicity. The character expansion of $\tilde{Z}_1$ at $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(2,0,0)$ can be written as

$$
\begin{aligned}
\tilde{Z}_1 = \frac{t^{18}}{(1-t^2)^{18}} \Big( & -96096\,(\chi^s_{(13)} + \chi^s_{(31)}) \cdot (7t^4 + 42t^2 + 72 + 42t^{-2} + 7t^{-4}) \\
& + 10010\,(\chi^s_{(24)} + \chi^s_{(42)}) \cdot (9t^5 + 88t^3 + 243t + 243t^{-1} + 88t^{-3} + 9t^{-5}) \\
& - 352\,(\chi^s_{(15)} + \chi^s_{(51)}) \cdot (25t^6 + 474t^4 + 2169t^2 + 3504 + 2169t^{-2} + 474t^{-4} + 25t^{-6}) \\
& - 2464\,(\chi^s_{(35)} + \chi^s_{(53)}) \cdot (2t^6 + 27t^4 + 108t^2 + 168 + 108t^{-2} + 27t^{-4} + 2t^{-6}) \\
& + 11\,(\chi^s_{(06)} + \chi^s_{(60)}) \cdot (42t^7 + 1194t^5 + 8451t^3 + 21253t + 21253t^{-1} + \cdots + 42t^{-7}) \\
& + 11\,(\chi^s_{(26)} + \chi^s_{(62)}) \cdot (45t^7 + 1101t^5 + 6983t^3 + 16623t + 16623t^{-1} + \cdots + 45t^{-7}) \\
& - 32\,(\chi^s_{(17)} + \chi^s_{(71)}) \cdot (t^8 + 36t^6 + 336t^4 + 1176t^2 + 17641 + 1176t^{-2} + \cdots + t^{-8}) \\
& + 99\,\chi^s_{(22)} \cdot (5t^9 - 90t^7 + 1623t^5 + 26743t^3 + 83103t + (t \to t^{-1})) \\
& + 462\,(\chi^s_{(02)} + \chi^s_{(20)}) \cdot (t^9 - 18t^7 + 153t^5 + 4059t^3 + 13485t + (t \to t^{-1})) \\
& - 32\,\chi^s_{(33)} \cdot (t^{10} - 18t^8 + 450t^6 + 13340t^4 + 66977t^2 + 110772 + 66977t^{-2} + \cdots + t^{-10}) \\
& + \chi^s_{(44)} \cdot (t^{11} - 18t^9 + 615t^7 + 26332t^5 + 187749t^3 + 466001t^1 + 466001t^{-1} + \cdots + t^{-11}) \\
& + (\chi^s_{(04)} + \chi^s_{(40)}) \cdot (t^{13} - 18t^{11} + 153t^9 - 816t^7 + 58115t^5 + 730170t^3 + 2129595t^1 + (t \to t^{-} \\
& - 352\,\chi^s_{(11)} \cdot (t^{10} - 4t^8 - 99t^6 + 2496t^4 + 18246t^2 + 32976 + 18246t^{-2} + \cdots + t^{-10}) \\
& + (\chi^s_{(08)} + \chi^s_{(80)}) \cdot (t^9 + 48t^7 + 603t^5 + 2898t^3 + 6174t + (t \to t^{-1})) \\
& + (t^{17} - 18t^{15} + 153t^{13} - 739t^{11} + 3753t^9 - 20195t^7 + 49881t^5 + 1203597t^3 + 4481279t^1 + (
\end{aligned}
\tag{A.0.11}
$$

It was explicitly checked that the 1-instanton partition function $Z_1$ at $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (0,2,0)$ could be identified with the above as

$$
Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(0,2,0)}(a_1,a_2,a_3,a_4,a_5,a_6) = Z_1^{(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}})=(2,0,0)}(a_1,a_2,a_3,a_4,a_5,-a_6). \tag{A.0.12}
$$

Similarly, the character expansion of $\tilde{Z}_1$ at $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (1,1,0)$ can be displayed as follows:

$$
\begin{aligned}
\tilde{Z}_1 = \frac{t^{18}}{(1-t^2)^{18}} \sum_{\pm} \Big( & - 2462\, (y_s^{\pm 1} y_c^{\pm 4} + y_s^{\pm 4} y_c^{\pm 1}) \cdot (2t^6 + 27t^4 + 108t^2 + 168 + 108t^{-2} + 27t^{-4} - \\
& + 11\, (y_s^{\pm 2} y_c^{\pm 4} + y_s^{\pm 4} y_c^{\pm 2}) \cdot (45t^7 + 1101t^5 + 6983t^3 + 16623t + (t \to t^{-1})) \\
& + 44\, (y_s^{\pm 3} y_c^{\pm 3}) \cdot (23t^7 + 587t^5 + 3925t^3 + 9609t + (t \to t^{-1})) \qquad \text{(A.0.13)} \\
& + 44\, (y_s^{\pm 1} y_c^{\pm 3} + y_s^{\pm 3} y_c^{\pm 1}) \cdot (23t^7 + 2927t^5 + 26025t^3 + 70033t + (t \to t^{-1})) \\
& - 32\, (y_s^{\pm 3} y_c^{\pm 4} + y_s^{\pm 4} y_c^{\pm 3}) \cdot (t^8 + 36t^6 + 336t^4 + 1176t^2 + 1764 + 1176t^{-2} + \cdots + t^{-8}) \\
& - 32\, (y_s^{\pm 2} y_c^{\pm 3} + y_s^{\pm 3} y_c^{\pm 2}) \cdot (t^8 + 465t^6 + 7629t^4 + 33351t^2 + 53244 + 33351t^{-2} + \cdots + t^{-8}) \\
& + (y_s^{\pm 4} y_c^{\pm 4}) \cdot (t^9 + 48t^7 + 603t^5 + 2898t^3 + 6174t + (t \to t^{-1})) \\
& - 32\, (y_s^{\pm 1} y_c^{\pm 2} + y_s^{\pm 2} y_c^{\pm 1}) \cdot (t^{10} - 17t^8 + 1069t^6 + 44069t^4 + 234770t^2 + 393168 + 234770t^{-2} \\
& - 32\, (y_s^{\pm 3} + y_c^{\pm 3}) \cdot (t^{10} - 17t^8 + 750t^6 + 17526t^4 + 83553t^2 + 136714 + 83358t^{-2} + \cdots + t^{-1} \\
& - 32\, (y_s^{\pm 1} + y_c^{\pm 1}) \cdot (13t^{10} - 79t^8 + 408t^6 + 97724t^4 + 587351t^2 + 1011546 + 587351t^{-2} + \cdots \\
& + (y_s^{\pm 4} + y_c^{\pm 4}) \cdot (t^{11} - 18t^9 + 615t^7 + 26332t^5 + 187749t^3 + 466001t + (t \to t^{-1})) \\
& + (y_s^{\pm 2} + y_c^{\pm 2}) \cdot (t^{11} + 477t^9 - 7305t^7 + 391411t^5 + 4750692t^3 + 13923764t^1 + (t \to t^{-1})) \\
& + 4\, (y_s^{\pm 1} y_c^{\pm 1}) \cdot (3t^{11} + 199t^9 + 132676t^7 + 1864041t^5 + 5630341t^3 + 5630341t^{+}(t \to t^{-1})) \\
& + (y_s^{\pm 2} y_c^{\pm 2})(t^{13} - 17t^{11} + 1136t^9 + 804t^7 + 200385t^5 + 1971471t^3 + 5450836t + (t \to t^{-1})) \\
& + (t^{15} - 17t^{13} + 214t^{11} + 1414t^9 - 33152t^7 + 704404t^5 + 11381979t^3 + 35592757t + (t \to t^{-}
\end{aligned}
$$

in which $\sum_{\pm}$ notation is understood as follows: $\sum_{\pm} x^{\pm 1} y^{\pm 1} = xy + xy^{-1} + x^{-1}y + x^{-1}y^{-1}$, $\sum_{\pm} x^{\pm 1} = x + x^{-1}$, and $\sum_{\pm} 1 = 1$.

**SO(13)** The flavor symmetry on $N_{\mathbf{s}}\mathbf{S} + N_{\mathbf{v}}\mathbf{V}$ matter multiplets is $SO(2N_{\mathbf{s}})_s \times Sp(N_{\mathbf{v}})_v$. The character expansion of $\tilde{Z}_1$ at $(N_{\mathbf{s}}, N_{\mathbf{v}}) = (1,1)$ is written follows, after setting the Coulomb VEV $\vec{a} = 0$ to keep the expression concise,

$$
\begin{aligned}
\tilde{Z}_1 = \frac{t^{20}}{(1-t^2)^{20}} \sum_{\pm} \Big( & y_s^{\pm 8} \chi^V_{(1)} \cdot (t^{10} + 58t^8 + 905t^6 + 5580t^4 + 15876t^2 + 22344 + +15876t^{-2} \cdots \\
& - 64\, y_s^{\pm 7} \chi^V_{(1)} \cdot (t^9 + 45t^7 + 540t^5 + 2520t^3 + 5292t + (t \to t^{-1})) \\
& + 26\, y_s^{\pm 6} \chi^V_{(1)} \cdot (77t^8 + 2541t^6 + 22226t^4 + 74811t^2 + 110770 + 74811t^{-2} + \cdots + 77t^{-8}) \\
& - 5824\, y_s^{\pm 5} \chi^V_{(1)} \cdot (7t^7 + 154t^5 + 924t^3 + 2145t + (t \to t^{-1})) \\
& + y_s^{\pm 4} \chi^V_{(1)} \cdot (t^{14} - 19t^{12} + 170t^{10} + 766t^8 + 576628t^6 + 7601283t^4 + 29870761t^2 \\
& \qquad + 46175700 + 29870761t^{-2} + 7601283t^{-4} + \cdots + \\
& - 64\, y_s^{\pm 3} \chi^V_{(1)} \cdot (t^{11} - 20t^9 + 1256t^7 + 83074t^5 + 628311t^3 + 1580032t + (t \to t^{-1})) \\
& + 2002\, y_s^{\pm 2} \chi^V_{(1)} \cdot (t^{10} - 19t^8 + 756t^6 + 15006t^4 + 66051t^2 + 105146 + 66051t^{-2} + \cdots + t^{-10}) \\
& - 64\, y_s^{\pm 1} \chi^V_{(1)} \cdot (13t^{11} - 51t^9 - 436t^7 + 182670t^5 + 1603925t^3 + 4218449t + (t \to t^{-1})) \\
& + \chi^V_{(1)} \cdot (t^{16} - 19t^{14} + 274t^{12} + 3185t^{10} - 73808t^8 + 1918679t^6 + 46355974t^4 + 212905247t^2 \\
& \qquad + 342439014 + 212905247t^{-2} + 46355974t^{-4} + \cdots + \\
& - 13\, y_s^{\pm 8} \cdot (t^9 + 35t^7 + 365t^5 + 1575t^3 + 3192t + (t \to t^{-1})) \\
& + 256\, y_s^{\pm 7} \cdot (3t^8 + 80t^6 + 630t^4 + 2016t^2 + 2940 + 2016t^{-2} + \cdots + 3t^{-8}) \\
& - 26\, y_s^{\pm 6} \cdot (847t^7 + 15989t^5 + 89887t^3 + 203357t + (t \to t^{-1})) \\
& - 64\, y_s^{\pm 5} \cdot (t^{10} - 20t^8 - 6180t^6 - 75228t^4 - 286725t^2 - 439416 - 286725t^{-2} + \cdots + t^{-10}) \\
& + y_s^{\pm 4} \cdot (t^{13} - 20t^{11} + 2907t^9 - 74785t^7 - 4557934t^5 - 33690015t^3 - 83955034t + (t \to t^{-1})) \\
& - 64\, y_s^{\pm 3} \cdot (t^{12} - 19t^{10} + 807t^8 - 24636t^6 - 510121t^4 - 2255129t^2 - 3592422 \\
& \qquad - 2255129t^{-2} - 510121t^{-4} - 24636t^{-6} + \cdots + t^{-12}) \\
& + 2\, y_s^{\pm 2} \cdot (7t^{13} - 140t^{11} + 3189t^9 + 86972t^7 - 7685485t^5 - 71293018t^3 - 190116261t + (t \to \\
& - 64\, y_s^{\pm 1} \cdot (t^{12} - 84t^{10} + 2667t^8 - 36526t^6 - 1227485t^4 - 5926190t^2 - 9643046 \\
& \qquad - 5926190t^{-2} - 1227485t^{-4} - 36526t^{-6} + \cdots + t^{-12}
\end{aligned}
$$

$$+ (t^{15} - 20t^{13} - 602t^{11} + 5691t^{9} + 495005t^{7} - 22183672t^{5} - 225823570t^{3} - 617150913t + (t$$

(A.0.14)

**SO(14)** The classical flavor symmetry on $N_{\mathbf{s}}\mathbf{S}+N_{\mathbf{c}}\mathbf{C}+N_{\mathbf{v}}\mathbf{V}$ hypermultiplets is $U(N_{\mathbf{s}})_s\times U(N_{\mathbf{c}})_c\times Sp(N_{\mathbf{v}})_v$. The character expansion of $\tilde{Z}_1$ at $(N_{\mathbf{s}},N_{\mathbf{c}},N_{\mathbf{v}}) = (1,0,2)$ is written as follows, after turning off the $SO(14)$ Coulomb VEV $\vec{a}=0$,

$$\tilde{Z}_1 = \frac{t^{22}}{(1-t^2)^{22}}\sum_{\pm}\Big(y_s^{\pm 8}\chi^V_{(01)}\cdot(t^{11}+69t^9+1309t^7+10065t^5+36828t^3+69300t+(t\to t^{-1}))$$

$$- 64\, y_s^{\pm 7}\chi^V_{(01)}\cdot(t^{10}+55t^8+825t^6+4950t^4+13860t^2+19404+13860t^{-2}+\cdots+t^{-10})$$

$$+ 26\, y_s^{\pm 6}\chi^V_{(01)}\cdot(77t^9+3234t^7+36667t^5+164401t^3+338261t+(t\to t^{-1}))$$

$$- 5824\, y_s^{\pm 5}\chi^V_{(01)}\cdot(7t^8+210t^6+1694t^4+5434t^2+7920+5434t^{-2}+\cdots+7t^{-8})$$

$$+ y_s^{\pm 4}\chi^V_{(01)}\cdot(t^{15}-22t^{13}+231t^{11}-1540t^9+614558t^7$$

$$+ 11510191t^5+62671224t^3+139186397t+(t\to t^-$$

$$- 832\, y_s^{\pm 3}\chi^V_{(01)}\cdot(33t^8+7744t^6+83776t^4+300104t^2+451192+300104t^{-2}+\cdots+33t^{-8})$$

$$+ 2002\, y_s^{\pm 2}\chi^V_{(01)}\cdot(t^{11}-22t^9+621t^7+21262t^5+134245t^3+314181t+(t\to t^{-1}))$$

$$- 832\, y_s^{\pm 1}\chi^V_{(01)}\cdot(t^{12}-t^{10}-231t^8+15631t^6+206987t^4+790240t^2+1207976+$$

$$+ 790240t^{-2}+206987t^{-4}+15631t^{-6}+\cdots+t^{-12})$$

$$+ \chi^V_{(01)}\cdot(t^{17}-22t^{15}+335t^{13}+3179t^{11}-84595t^9+1320011t^7$$

$$+ 63966077t^5+427850621t^3+1020096033t+(t\to$$

$$- 14\, y_s^{\pm 8}\chi^V_{(10)}\cdot(t^{10}+42t^8+539t^6+2948t^4+7854t^2+10824+7854t^{-2}+\cdots+t^{-10})$$

$$+ 832\, y_s^{\pm 7}\chi^V_{(10)}\cdot(t^9+33t^7+330t^5+1386t^3+2772t+(t\to t^{-1}))$$

$$- 2184\, y_s^{\pm 6}\chi^V_{(10)}\cdot(11t^8+270t^6+2002t^4+6182t^2+8910+6182t^{-2}+\cdots+11t^{-8})$$

$$+ 5824\, y_s^{\pm 5}\chi^V_{(10)}\cdot(77t^7+1281t^5+6677t^3+14575t+(t\to t^{-1}))$$

$$+ 52\, y_s^{\pm 4}\chi^V_{(10)}\cdot(33t^{10}-726t^8-109153t^6-1133396t^4-3996580t^2-5980436$$

$$- 3996580t^{-2}-1133396t^{-4}-109153t^{-6}+\cdots+33t$$

$$
\begin{aligned}
&- 64\, y_s^{\pm 3} \chi^V_{(10)} \cdot (t^{13} - 22t^{11} + 868t^9 - 20559t^7 - 726341t^5 - 4583956t^3 - 10718569t + (t \to t \\
&+ 8008\, y_s^{\pm 2} \chi^V_{(10)} \cdot (49t^8 - 2102t^6 - 29678t^4 - 115094t^2 - 176638 - 115094t^{-2} + \cdots + 49t^{-8}) \\
&+ 4928\, y_s^{\pm 1} \chi^V_{(10)} \cdot (t^{11} - 35t^9 + 217t^7 + 21505t^5 + 152866t^3 + 371316t + (t \to t^{-1})) \\
&- 8\, \chi^V_{(10)} \cdot t^{10}(112t^{12} - 189t^{10} - 104258t^8 + 2855160t^6 + 46213090t^4 + 185620270t^2 \\
&\qquad\qquad\qquad\qquad + 287407450 + 185620270t^{-2} + 46213090t^{-4} + \cdots + \\
&+ 13\, y_s^{\pm 8} \cdot (8t^9 + 229t^7 + 2101t^5 + 8393t^3 + 16401t + (t \to t^{-1})) \\
&- 5824\, y_s^{\pm 7} \cdot (t^8 + 22t^6 + 154t^4 + 462t^2 + 660 + 462t^{-2} + \cdots + t^{-8}) \\
&+ 26\, y_s^{\pm 6} \cdot (6075t^7 + 95425t^5 + 483483t^3 + 1042937t + (t \to t^{-1})) \\
&- 64\, y_s^{\pm 5} \cdot (t^{12} - 22t^{10} + 231t^8 + 41580t^6 + 427575t^4 + 1498244t^2 + 2237312 \\
&\qquad\qquad\qquad\qquad + 1498244t^{-2} + 427575t^{-4} + 41580t^{-6} + \cdots + t^{-12}) \\
&+ 91\, y_s^{\pm 4} \cdot (11t^{11} - 473t^9 + 7623t^7 + 312675t^5 + 2010490t^3 + 4723994t + (t \to t^{-1})) \\
&+ 5824\, y_s^{\pm 3} \cdot (77t^8 - 2046t^6 - 32546t^4 - 129768t^2 - 200508 - 129768t^{-2} + \cdots + 77t^{-8}) \\
&+ 2\, y_s^{\pm 2} \cdot (7t^{15} - 154t^{13} + 2475t^{11} - 93720t^9 - 257649t^7 \\
&\qquad\qquad\qquad\qquad + 50128782t^5 + 390072133t^3 + 972422990t + (t \to t \\
&- 64\, y_s^{\pm 1} \cdot (t^{14} - 22t^{12} + 231t^{10} - 24927t^8 + 317625t^6 + 7227990t^4 + 31070743t^2 + 4891268 \\
&\qquad\qquad\qquad\qquad + 31070743t^{-2} + 7227990t^{-4} + 317625t^{-6} + \cdots + t^{-} \\
&+ 154\,(20t^{11} - 1740t^9 - 16109t^7 + 958563t^5 + 8046291t^3 + 20489955t + (t \to t^{-1})).
\end{aligned}
\tag{A.0.15}
$$

We also confirmed that the 1-instanton partition function $Z_1$ for $(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}}) = (0, 1, 2)$ could be identified with the above as follows:

$$
Z_1^{(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}})=(0,1,2)}(\vec{a}, \epsilon_1, \epsilon_2, m^c, \vec{m}^v) = Z_1^{(N_{\mathbf{s}}, N_{\mathbf{c}}, N_{\mathbf{v}})=(1,0,2)}(\vec{a}, \epsilon_1, \epsilon_2, m^s, \vec{m}^v)|_{m^s \to -m^c}.
\tag{A.0.16}
$$

**$\mathbf{E_6}$** The flavor symmetry on $N_f\mathbf{F}+N_{\bar f}\bar{\mathbf{F}}$ hypermultiplets is $U(N_f+N_{\bar f})$. The character expansion of $\tilde{Z}_1$ at $N_f+N_{\bar f}=3$ is written as follows:

$$\begin{aligned}
\tilde{Z}_1 = \frac{t^{22}}{(1-t^2)^{22}} \Big( &(\chi^f_{(00)_{-9}}+\chi^f_{(00)_9})(t^{11}+56t^9+945t^7+6776t^5+23815t^3+43989t+(t\to t^{-1}) \\
&-27(\chi^f_{(10)_{-8}}+\chi^f_{(01)_8})(t^{10}+42t^8+539t^6+2948t^4+7854t^2+10824+7854t^{-2}+\cdots+t^{-10} \\
&+351(\chi^f_{(20)_{-7}}+\chi^f_{(02)_7})(t^9+28t^7+253t^5+1001t^3+1947t+(t\to t^{-1})) \\
&+351(\chi^f_{(01)_{-7}}+\chi^f_{(10)_7})(t^9+33t^7+330t^5+1386t^3+2772t+(t\to t^{-1}))
\end{aligned} \tag{A.0.17}$$

$$\begin{aligned}
&+(\chi^f_{(30)_{-6}}+\chi^f_{(03)_6})(t^{12}-22t^{10}-2694t^8-42790t^6-256355t^4 \\
&\qquad\qquad -712536t^2-994488-712536t^{-2}+\cdots+t \\
&-26(\chi^f_{(11)_{-6}}+\chi^f_{(11)_6})(224t^8+4774t^6+32700t^4+96877t^2+137830+96877t^{-2}+\cdots+224 \\
&-13(\chi^f_{(00)_{-6}}+\chi^f_{(00)_6})(231t^8+6182t^6+48796t^4+156338t^2+228074+156338t^{-2}+\cdots+2 \\
&+351(\chi^f_{(40)_{-5}}+\chi^f_{(04)_5})(t^9+28t^7+253t^5+1001t^3+1947t+(t\to t^{-1})) \\
&-27(\chi^f_{(21)_{-5}}+\chi^f_{(12)_5})(t^{11}-22t^9-1694t^7-19965t^5-89298t^3-182952t+(t\to t^{-1})) \\
&+702(\chi^f_{(02)_{-5}}+\chi^f_{(20)_5})(49t^7+707t^5+3399t^3+7150t+(t\to t^{-1})) \\
&+702(\chi^f_{(10)_{-5}}+\chi^f_{(01)_5})(77t^7+1281t^5+6677t^3+14575t+(t\to t^{-1})) \\
&-27(\chi^f_{(50)_{-4}}+\chi^f_{(05)_4})(t^{10}+42t^8+539t^6+2948t^4+7854t^2+10824+7854t^{-2}+\cdots t^{-10}) \\
&-351(\chi^f_{(31)_{-4}}+\chi^f_{(13)_4})(21t^8+434t^6+2926t^4+8602t^2+12210+8602t^{-2}+\cdots+21t^{-8}) \\
&+351(\chi^f_{(12)_{-4}}+\chi^f_{(21)_4})(t^{10}-22t^8-869t^6-6908t^4-21714t^2-31416-21714t^{-2}+\cdots+t \\
&+351(\chi^f_{(20)_{-4}}+\chi^f_{(02)_4})(t^{10}-22t^8-1177t^6-10500t^4-34936t^2-51436-34936t^{-2}+\cdots+ \\
&-1404(\chi^f_{(01)_{-4}}+\chi^f_{(10)_4})(294t^6+3132t^4+10989t^2+16390+10989t^{-2}+3132t^{-4}+294t^{-6}) \\
&+(\chi^f_{(60)_{-3}}+\chi^f_{(06)_3})(t^{11}+56t^9+945t^7+6776t^5+23815t^3+43989t+(t\to t^{-1})) \\
&+13(\chi^f_{(41)_{-3}}+\chi^f_{(14)_3})(50t^9+1573t^7+15219t^5+62623t^3+124025t+(t\to t^{-1})) \\
&+(\chi^f_{(22)_{-3}}+\chi^f_{(22)_3})(t^{13}-22t^{11}+231t^9+68530t^7+919589t^5
\end{aligned}$$

$$+ 4310670t^3 + 8985999t + (t \to t^{-1}))$$

$$- 13(\chi^f_{(03)_{-3}} + \chi^f_{(30)_3})(6t^{11} + 93t^9 - 3564t^7 - 60115t^5 - 303171t^3 - 650699t + (t \to t^{-1}))$$

$$+ 13(\chi^f_{(30)_{-3}} + \chi^f_{(03)_3})(6075t^7 + 95425t^5 + 483483t^3 + 1042937t + (t \to t^{-1}))$$

$$- 832(\chi^f_{(11)_{-3}} + \chi^f_{(11)_3}(7t^9 - 154t^7 - 4095t^5 - 23683t^3 - 53471t + (t \to t^{-1}))$$

$$+ (\chi^f_{(00)_{-3}} + \chi^f_{(00)_3})(t^{15} - 22t^{13} + 231t^{11} - 1540t^9 + 7315t^7$$

$$+ 1533042t^5 + 10536141t^3 + 24960012t + (t$$

$$- 27(\chi^f_{(51)_{-2}} + \chi^f_{(15)_2})(t^{10} + 42t^8 + 539t^6 + 2948t^4 + 7854t^2 + 10824 + 7854t^{-2} + \cdots + t^{-10}$$

$$- 351(\chi^f_{(32)_{-2}} + \chi^f_{(23)_2})(21t^8 + 434t^6 + 2926t^4 + 8602t^2 + 12210 + 8602t^{-2} + \cdots + 21t^{-8})$$

$$+ 351(\chi^f_{(13)_{-2}} + \chi^f_{(31)_2})(t^{10} - 22t^8 - 869t^6 - 6908t^4 - 21714t^2 - 31416 - 21714t^{-2} + \cdots + t$$

$$- 702(\chi^f_{(40)_{-2}} + \chi^f_{(04)_2})(11t^8 + 270t^6 + 2002t^4 + 6182t^2 + 8910 + 6182t^{2-} + \cdots + 11t^{-8})$$

$$- 27(\chi^f_{(21)_{-2}} + \chi^f_{(12)_2})(t^{12} - 22t^{10} + 231t^8 + 34300t^6 + 334235t^4$$

$$+ 1139314t^2 + 1686762 + 1139314t^{-2} + \cdots$$

$$+ 27(\chi^f_{(02)_{-2}} + \chi^f_{(20)_2})(64t^{10} + 517t^8 - 27566t^6 - 317548t^4 - 1145354t^2$$

$$- 1723106 - 1145354t^{-2} + \cdots + 64t^{-10})$$

$$+ 4914(\chi^f_{(10)_{-2}} + \chi^f_{(01)_2})(7t^8 - 154t^6 - 2310t^4 - 8866t^2 - 13530 - 8866t^{-2} + \cdots + 7t^{-8})$$

$$+ 351(\chi^f_{(42)_{-1}} + \chi^f_{(24)_1})(t^9 + 28t^7 + 253t^5 + 1001t^3 + 1947t + (t \to t^{-1}))$$

$$+ 351(\chi^f_{(50)_{-1}} + \chi^f_{(05)_1})(t^9 + 33t^7 + 330t^5 + 1386t^3 + 2772t + (t \to t^{-1}))$$

$$- 27(\chi^f_{(23)_{-1}} + \chi^f_{(32)_1})(t^{11} - 22t^9 - 1694t^7 - 19965t^5 - 89298t^3 - 182952t + (t \to t^{-1}))$$

$$+ 22464(\chi^f_{(31)_{-1}} + \chi^f_{(13)_1})(5t^7 + 77t^5 + 385t^3 + 825t + (t \to t^{-1}))$$

$$+ 702(\chi^f_{(04)_{-1}} + \chi^f_{(40)_1})(49t^7 + 707t^5 + 3399t^3 + 7150t + (t \to t^{-1}))$$

$$- 351(\chi^f_{(12)_{-1}} + \chi^f_{(21)_1})(21t^9 - 462t^7 - 11605t^5 - 65983t^3 - 148071t + (t \to t^{-1}))$$

$$+ 351(\chi^f_{(20)_{-1}} + \chi^f_{(02)_1})(t^{11} - 22t^9 + 231t^7 + 11516t^5 + 72799t^3 + 168707t + (t \to t^{-1}))$$

$$- 702(\chi^f_{(01)_{-1}} + \chi^f_{(10)_1})(25t^9 - 6325t^5 - 44583t^3 - 107387t + (t \to t^{-1}))$$

$$
\begin{aligned}
&+ \chi^f_{(33)_0}(t^{12} - 22t^{10} - 2694t^8 - 42790t^6 - 256355t^4 - 712536t^2 - 994488 - 712536t^{-2} + \cdots \\
&- 26(\chi^f_{(41)_0} + \chi^f_{(14)_0})(224t^8 + 4774t^6 + 32700t^4 + 96877t^2 + 137830 + 96877t^{-2} + \cdots + t^{-8}) \\
&+ 26\,\chi^f_{(22)_0}(25t^{10} - 550t^8 - 27030t^6 - 231990t^4 - 756657t^2 - 1107156 - 756657t^{-2} + \cdots + 2 \\
&+ (\chi^f_{(03)_0} + \chi^f_{(30)_0})(t^{14} - 22t^{12} + 231t^{10} - 1540t^8 - 593285t^6 - 5973198t^4 \\
&\qquad\qquad\qquad\qquad - 20531379t^2 - 30453456 - 20531379t^{-2} + \\
&- 26\,\chi^f_{(11)_0}(3t^{12} - 66t^{10} - 2002t^8 + 54670t^6 + 741975t^4 + 2786872t^2 \\
&\qquad\qquad\qquad\qquad + 4232536 + 2786872t^{-2} + \cdots + 3t^{-12}) \\
&+ 2(1215t^{10} + 26070t^8 - 212410t^6 - 4381850t^4 - 18219943t^2 \\
&\qquad\qquad\qquad\qquad - 28496524 - 18219943t^{-2} + \cdots + 1215t^{-1
\end{aligned}
$$

where the $U(3)$ character $\chi^f_{(mn)_c}$ is defined as

$$
\chi^f_{(mn)_c} = (w_1 w_2 w_3)^{\frac{c-m+n}{3}} \left( \sum_{\substack{1 \le i_1 \le \cdots \le i_m \le 3 \\ 1 \le j_1 \le \cdots \le j_n \le 3}} \frac{w_{i_1} \cdots w_{i_m}}{w_{j_1} \cdots w_{j_n}} - \sum_{\substack{1 \le i_1 \le \cdots \le i_{m-1} \le 3 \\ 1 \le j_1 \le \cdots \le j_{n-1} \le 3}} \frac{w_{i_1} \cdots w_{i_{m-1}}}{w_{j_1} \cdots w_{j_{n-1}}} \right), \tag{A.0.18}
$$

with

$$
(w_1, w_2, w_3) = \begin{cases} (y_{f,1}, y_{f,2}, y_{f,3}) & \text{for} \quad (N_f, N_{\bar{f}}) = (3,0), \\ (y_{f,1}, y_{f,2}, y^{-1}_{\bar{f},1}) & \text{for} \quad (N_f, N_{\bar{f}}) = (2,1), \\ (y_{f,1}, y^{-1}_{\bar{f},1}, y^{-1}_{\bar{f},2}) & \text{for} \quad (N_f, N_{\bar{f}}) = (1,2), \\ (y^{-1}_{\bar{f},1}, y^{-1}_{\bar{f},2}, y^{-1}_{\bar{f},3}) & \text{for} \quad (N_f, N_{\bar{f}}) = (0,3). \end{cases} \tag{A.0.19}
$$

Again, (A.0.17) was tested against our general 1-instanton expression (5.1.39) up to $t^{180}$ order.

**$\mathbf{E_7}$** The flavor symmetry acting on $N_f$ hypermultiplets is $SO(2N_f)_f$. The character expansion of $\tilde{Z}_1$ at $N_f = 2$ is given as follows:

$$\tilde{Z}_1 = \frac{t^{34}}{(1-t^2)^{34}}\Big((\chi^f_{(0,12)} + \chi^f_{(12,0)})(t^{17} + 99t^{15} + 3410t^{13} + 56617t^{11} + 521917t^9 \tag{A.0.20}$$

$$+ 2889898t^7 + 10086066t^5 + 22867856t^3 + 34289476t + (t \to t^{-1})$$

$$- 8(\chi^f_{(1,11)} + \chi^f_{(11,1)})(7t^{16} + 572t^{14} + 16401t^{12} + 227766t^{10} + 1759296t^8$$

$$+ 8155308t^6 + 23747878t^4 + 44652608t^2 + 55026348 + 44652608t^{-2} + \cdots + 7$$

$$+ 19(\chi^f_{(2,10)} + \chi^f_{(10,2)})(81t^{15} + 5254t^{13} + 121550t^{11} + 1376580t^9$$

$$+ 8725369t^7 + 33273284t^5 + 79629972t^3 + 122510670t + (t \to t^{-1}))$$

$$+ 133(\chi^f_{(0,10)} + \chi^f_{(10,0)})(11t^{15} + 760t^{13} + 18445t^{11} + 216580t^9$$

$$+ 1409980t^7 + 5479474t^5 + 13273260t^3 + 20541950t + (t \to t^{-1}))$$

$$- 152(\chi^f_{(39)} + \chi^f_{(93)})(182t^{14} + 8827t^{12} + 158592t^{10} + 1426827t^8 + 7281032t^6$$

$$+ 22506946t^4 + 43735356t^2 + 54466776 + 43735356t^{-2} + \cdots + 182t^{-14})$$

$$- 2128(\chi^f_{(19)} + \chi^f_{(91)})(24t^{14} + 1309t^{12} + 25454t^{10} + 241859t^8$$

$$+ 1281324t^6 + 4059022t^4 + 7997752t^2 + 10005112 + 7997752t^{-2} + \cdots + 24t$$

$$+ (\chi^f_{(48)} + \chi^f_{(84)})(t^{19} - 34t^{17} + 561t^{15} + 359766t^{13} + 11997546t^{11} + 161435604t^9$$

$$+ 1130192844t^7 + 4579505424t^5 + 11356618494t^3 + 17763983094t + (t \to t^{-}$$

$$+ 10773(\chi^f_{(28)} + \chi^f_{(82)})(91t^{13} + 3668t^{11} + 54893t^9$$

$$+ 411026t^7 + 1739100t^5 + 4427038t^3 + 7011004t + (t \to t^{-1}))$$

$$+ 5187(\chi^f_{(08)} + \chi^f_{(80)})(119t^{13} + 5269t^{11} + 83499t^9$$

$$+ 648329t^7 + 2806870t^5 + 7243122t^3 + 11543952t + (t \to t^{-1}))$$

$$+ 912(\chi^f_{(57)} + \chi^f_{(75)})(t^{16} - 34t^{14} - 3597t^{12} - 78540t^{10} - 776832t^8 - 4186896t^6$$

$$- 13370126t^4 - 26439556t^2 - 33110220 - 26439556t^{-2} + \cdots + t^{-16})$$

$$- 56(\chi^f_{(37)} + \chi^f_{(73)})(t^{18} - 34t^{16} + 561t^{14} + 227392t^{12} + 6213449t^{10} + 69122350t^8 + 40017416$$

$$+ 1335305664t^4 + 2705039932t^2 + 3413732872 + 2705039932t^{-2} + \cdots + t^{-18}$$

$$- 27664(\chi^f_{(17)} + \chi^f_{(71)})(539t^{12} + 17314t^{10} + 208879t^8 + 1267860t^6$$

$$+ 4351490t^4 + 8949752t^2 + 11348792 + 8949752t^{-2} + \cdots + 539t^{-12})$$

$$- 19\, \chi^f_{(66)}(7t^{17} + 217t^{15} - 24908t^{13} - 1021757t^{11} - 14769022t^9$$

$$- 107322042t^7 - 444417927t^5 - 1115908152t^3 - 1755535056t + (r \to t^{-1}))$$

$$- 95(\chi^f_{(46)} + \chi^f_{(64)})(429t^{15} - 14586t^{13} - 1157156t^{11} - 20646010t^9$$

$$- 168250530t^7 - 746606798t^5 - 1952106107t^3 - 3129466862t + (t \to t^{-1}))$$

$$+ 57(\chi^f_{(26)} + \chi^f_{(62)})(27t^{17} - 918t^{15} + 15147t^{13} + 3520341t^{11} + 75470346t^9$$

$$+ 672625723t^7 + 3141068903t^5 + 8453641548t^3 + 13732731903t + (t \to t^{-1}))$$

$$+ (\chi^f_{(06)} + \chi^f_{(60)})(t^{21} - 34t^{19} + 561t^{17} - 5984t^{15} + 46376t^{13} + 108842392t^{11} + 2613712872t^9$$

$$+ 24490191704t^7 + 117519798814t^5 + 321089011759t^3 + 525183176299t + (t$$

$$+ 1296\, \chi^f_{(55)}(5t^{16} + 110t^{14} - 14030t^{12} - 458914t^{10} - 5440765t^8 - 32547180t^6$$

$$- 110625460t^4 - 226279180t^2 - 286427492 - 226279180t^{-2} + \cdots + 5t^{-16})$$

$$+ 1064(\chi^f_{(35)} + \chi^f_{(53)})(810t^{14} - 27540t^{12} - 1538126t^{10} - 21575635t^8 - 140780490t^6$$

$$- 502663905t^4 - 1055162460t^2 - 1346539128 - 1055162460t^{-2} + \cdots + 810t^{-}$$

$$- 27664(\chi^f_{(15)} + \chi^f_{(51)})(t^{16} - 34t^{14} + 561t^{12} + 62832t^{10} + 1000416t^8$$

$$+ 6920904t^6 + 25507174t^4 + 54425228t^2 + 69808596 + 54425228t^{-2} + \cdots + t$$

$$+ \chi^f_{(44)}(t^{21} - 34t^{19} + 561t^{17} - 158136t^{15} - 1922955t^{13} + 320810876t^{11} + 7970822266t^9$$

$$+ 74975208858t^7 + 359889450611t^5 + 983025661861t^3 + 1607508212091t + ($$

$$- 133(\chi^f_{(24)} + \chi^f_{(42)})(t^{19} - 34t^{17} + 561t^{15} + 79101t^{13} - 2846514t^{11} - 102197931t^9$$

$$- 1080814746t^7 - 5500823076t^5 - 15503708076t^3 - 25710027486t + (t \to t^{-}$$

$$+ 665(\chi^f_{(04)} + \chi^f_{(40)})(13t^{17} + 108t^{15} - 11407t^{13} + 230758t^{11} + 11122199t^9$$

$$
\begin{aligned}
&\qquad + 125832753t^7 + 660902603t^5 + 1893530023t^3 + 3162878730t + (t \to t^{-1})) \\
&+ 152\,\chi^f_{(33)}(6t^{18} - 204t^{16} + 18381t^{14} + 20306t^{12} - 21755987t^{10} - 387061196t^8 - 2796155121 \\
&\qquad - 10534894066t^4 - 22728127951t^2 - 29251476496 - 22728127951t^{-2} + \cdots + \\
&- 1064(\chi^f_{(13)} + \chi^f_{(31)})(81t^{16} - 2754t^{14} - 49181t^{12} + 2732444t^{10} + 59237424t^8 + 458851114t^6 \\
&\qquad + 1789977134t^4 + 3929114222t^2 + 5083736372 + 3929114222t^{-2} + \cdots + t^{-16} \\
&+ 81\,\chi^f_{(22)}(91t^{17} + 41t^{15} - 356609t^{13} + 2951795t^{11} + 247685515t^9 \\
&\qquad + 3029637009t^7 + 16451185429t^5 + 47931732849t^3 + 80650803640t + (t \to t \\
&- 1312311(\chi^f_{(02)} + \chi^f_{(20)})(14t^{13} - 17t^{11} - 7752t^9 \\
&\qquad - 103411t^7 - 581570t^5 - 1724208t^3 - 2923116t + (t \to t^{-1})) \\
&+ 304\,\chi^f_{(11)}(11960t^{14} + 343681t^{12} - 7234554t^{10} - 208524209t^8 - 1747615980t^6 \\
&\qquad - 7073563915t^4 - 15807799502t^2 - 20565064322 - 15807799502t^{-2} + \cdots + 1 \\
&+ (t^{23} - 34t^{21} + 561t^{19} - 5984t^{17} - 192226t^{15} - 11212452t^{13} - 46556642t^{11} + 4966300623t^9 \\
&\qquad + 73315010528t^7 + 427928422856t^5 + 1291626014327t^3 + 2206690491962t +
\end{aligned}
$$

This was tested against the closed-form expression (5.1.39) up to $t^{280}$ order.

# 초록

이 논문은 초대칭 게이지 이론에서 순간자의 분배함수를 계산하기 위한 새로운 방법론을 소개한다. 우선 게이지군이 $SU(N), SO(N), Sp(N)$의 고전적 단순 리 군인 경우엔 순간자의 모듈라이 공간을 유클리드 4차원 공간의 게이지 이론에 대하여 알려진 ADHM 작도를 끈이론을 바탕으로 확장하여 순간자(4차원), 순간자 입자(5차원) 혹은 끈(6차원) 위에서 정의되는 행렬모형, 1차원 혹은 2차원 게이지-시그마 모형으로 확장하여 이해할 수 있다.

게이지군이 고전적 단순 리 군으로 주어진 경우이거나, 물질 장이 게이지군에 대하여 특정한 방식으로 상호작용하고 있는 경우엔 이러한 ADHM 작도가 알려지지 않아 순간자의 모듈라이 공간을 이해하는데 어려움이 발생한다. 이들 중, 특정한 경우엔 이들의 고전적 부분군의 ADHM 작도를 확장하여 순간자의 모듈라이 공간을 오메가 배경 하에 쿨롱 가지에서 해석할 수 있다. 그리고 이를 바탕으로 국소화 기법을 통하여 순간자의 분배함수를 계산하고, 6차원 (1,0) 초등각장론의 '힉스불가능한 묶음'의 자기쌍대끈의 분배함수 계산에 응용한다.

한편으론 순간자가 위치하는 4차원 공간의 중심을 부풀리면 전체 분배함수가 만족해야하는 특정 관계식을 얻을 수 있는데, 이를 부풀리기 방정식이라고 부른다. 이 부풀리기 방정식을 이용하면 섭동적 분배함수만으로 일반적인 게이지군과 물질 장의 표현에 대한 순간자 분배함수를 각 순간자 단계에 따라 순차적으로 계산할 수 있다. 이 논문의 후반부에선 먼저 4차원 게이지 이론의 부풀리기 방정식을 유도하고, 이를 5차원에 확장함과 동시에 이들이 작동하는 범위를 조사하며, 이들을 이용하여 일반적인 게이지 군과 물질 장에 대한 순간자의 분배함수를 계산하는 방법을 소개한다.

**주요어**: 순간자, 초대칭, 등각장론, 분배 함수, 끈이론

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